Interpolation with Unequal Intervals

Divided Differences ⇒ If the values of a function $y = f(x)$ are given at the points $x_0, x_1, \cdots x_n$, which are not necessarily equally-spaced, then we define

First Divided Difference : The first divided difference of $y = f(x)$ for the arguments x_0 and x_1 is denoted by $f(x_0, x_1)$ or $\underset{x_1}{\Delta} f(x_0)$.

$$f(x_0, x_1) = \underset{x_1}{\Delta} f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Similarly

$$f(x_1, x_2) = \underset{x_2}{\Delta} f(x_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$f(x_{n-1}, x_n) = \underset{n}{\Delta} f(x_{n-1}) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

Second Divided Differences : The second divided difference of $y = f(x)$ for the arguments x_0, x_1 & x_2 is denoted by $f(x_0, x_1, x_2)$ or $\underset{x_1 x_2}{\Delta^2} f(x_0)$

$$f(x_0, x_1, x_2) = \underset{x_1 x_2}{\Delta^2} f(x_0) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

Similarly

$$f(x_1, x_2, x_3) = \underset{x_2 x_3}{\Delta^2} f(x_1) = \frac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1}$$

$$f[x_{n-2}, x_{n-1}, x_n) = \underset{x_{n-1}, x_n}{\Delta^2} f(x_{n-2}) = \frac{f(x_{n-1}, x_n) - f(x_{n-2}, x_{n-1})}{x_n - x_{n-1}}$$

n^{th} Divided Differences :

The n^{th} divided difference of $y = f(x)$ for the arguments $x_0, x_1 \cdots x_{n-1}$ and x_n is denoted and defined as

$$f(x_0, x_1 \cdots x_n) = \underset{x_1 \cdots x_n}{\Delta^n} f(x_0) = \frac{f(x_1 \cdots x_n) - f(x_0 \cdots x_{n-1})}{x_n - x_0}$$

Ex- Construct the divided difference table for the following table

x	0	1	2	4	5	6
$f(x)$	1	14	15	5	6	19

Solu:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	1	$\frac{14-1}{1-0} = 13$	$\frac{1-13}{2-0} = -6$	$\frac{-2-(-6)}{4-0} = 1$	0
1	14	$\frac{15-14}{2-1} = 1$	$\frac{-5-1}{4-1} = -2$	$\frac{2-(-2)}{5-1} = 1$	0
2	15	$\frac{5-15}{4-2} = -5$	$\frac{1-(-5)}{5-2} = 2$	$\frac{6-2}{6-2} = 1$	
4	5	$\frac{6-5}{5-4} = 1$	$\frac{13-1}{6-4} = 6$		
5	6	$\frac{19-6}{6-5} = 13$			
6	19				

Q1 ⇒ Find the third divided difference $f(3,4,5,6)$ where $f(x) = x^3 - x$

Solu: $f(3) = 24,\ f(4) = 60,\ f(5) = 120,\ f(6) = 210$

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
3	24	$\frac{60-24}{4-3} = 36$	$\frac{60-36}{5-3} = \frac{24}{2} = 12$	$\frac{15-12}{6-3} = 1$
4	60	$\frac{120-60}{5-4} = 60$	$\frac{90-60}{6-4} = \frac{30}{2} = 15$	
5	120	$\frac{210-120}{6-5} = 90$		
6	210			

The third divided difference is 1

(2) Show that $\Delta^2 x^3 = x+y+z$

Solu: →

x	$f(x)$	First Divided Difference $\Delta f(x)$	Second Divided Difference $\Delta^2 f(x)$
x	x^3	$\frac{y^3-x^3}{y-x} = y^2+x^2+yx$	$\frac{(z^2+y^2+zy)-(y^2+x^2+xy)}{z-x}$
y	y^3	$\frac{z^3-y^3}{z-y} = z^2+y^2+zy$	
z	z^3		

$$\frac{z^2+y^2+zy-y^2-x^2-xy}{z-x} = \frac{z^2-x^2+y(z-x)}{z-x}$$

$$\Rightarrow \frac{(z-x)(z+x)+y(z-x)}{z-x}$$

$$\Rightarrow \frac{(z-x)(z+x+y)}{z-x}$$

$$\Rightarrow x+y+z$$

Hence Second divided Difference

$$\Delta^2 f(x) = \Delta^2 x^3 = x+y+z$$

Related Q:

(1) Prove that the third divided difference with arguments x_0, x_1, x_2 & x_3 of the function $\frac{1}{x}$ is $(-1)^3 \frac{1}{x_0 x_1 x_2 x_3}$.

2) Find the third divided difference ~~table~~ with arguments 2, 4, 9, 10 of the function $f(x) = x^3 - 2x$