Posets:

A relation on a set S is called a partial order if it is
(1) reflexive (2) antisymmetric (3) transitive.

(1) $aRa \ \forall a \in S$

(2) aRb & $bRa \Rightarrow a = b$

(3) aRb & $bRc \Rightarrow aRc$.

partial order is denoted by $\preceq$ (This is not $\leq$)

$(S, \preceq) \rightarrow$ Partially ordered set or POSET

$a \preceq b \rightarrow$ a preceeds b or b succeeds a

Ex $(Z, \leq) \rightarrow$ partially ordered set or POSET as

(1) $a \leq a \ \forall a \in Z$ $\therefore$ Reflexive

(2) $a \leq b$ & $b \leq a \Rightarrow a = b$ $\therefore$ Antisymmetric

(3) $a \leq b$ & $b \leq c \Rightarrow a \leq c$ $\therefore$ Transitive.

Ex $(P(S), \subseteq) \rightarrow$ POSET. (verify it)

$P(S)$ → Power set of S

Ex $(Z, /)$ $m/n \rightarrow$ m divides n.

Is this a partially ordered set?

[No. as $a/(-a)$ and $(-a)/a$ but $a \neq -a$
i.e, not antisymmetric]

Note: $(Z^+, /)$ is POSET (verify it).

Comparability: If $a \preceq b$ or $b \preceq a$ then a & b are comparable elements of $(S, \preceq)$ else they are not comparable.

eg $(Z^+, /)$ $2/4$ $\therefore$ 2 & 4 are comparable
$2 \nmid 5$ $\therefore$ 2 & 5 are not comparable.

<u>Totally Ordered Set (Linearly ordered set)</u>: If every two elements of S are comparable, then poset $(S, \preceq)$ is called a totally ordered set.

This is also called a <u>chain</u>.

$(Z^+, /)$ is not totally ordered as $2 \nmid 5$ while $(Z, \leq)$ is a totally ordered set or chain.

<u>Immediate successor</u>: Let $(S, \preceq)$ be a poset and $x, y \in S$, then y is immediate successor of x if $x \prec y$ and $\nexists\, z \in S$ such that $x \prec z \prec y$.

<u>Immediate predecessor</u>: In above x is called immediate predecessor of y.

<u>Ex</u> (1) $(Z^+, /)$ ~~$2 \nmid 4$~~ $2/4$ & $2/6$ ...
2 is immediate predecessor of 4 and 4 is immediate successor of 2.

(2) $S = \{1, 2, 3\}$. Then $(P(S), \subseteq)$ is a poset.
Let $A = \{1\}$, $B = \{1, 3\}$, $C = \{1, 2, 3\} \in P(S)$
Now, $A \subseteq B$ & $A \subseteq C$ and $B \subseteq C$
$\therefore$ A is immediate predecessor of B
and B " " " " C
B is immediate successor of A
& C " " " " B

Hasse Diagrams: A poset $(S, \preceq)$ can be represented by means of a diagram called Hasse diagram.

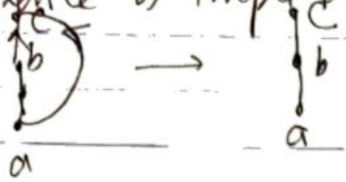
Here, if $x \preceq y$ then y is placed at a higher level than x and joined by a straight line.

steps:

1. Start with a directed graph of relation.
2. Remove the loop at all the vertices
3. If aRb, then b appears above a and is connected by an edge with arrow upwards.

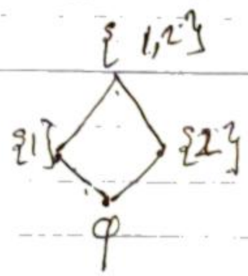
(4) Remove the arrows and all edges whose existence is implied by transitive property

e.g



Ex: $S = \{1, 2\}$. $P(S) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$

Then $(P(S), \subseteq)$ is a poset

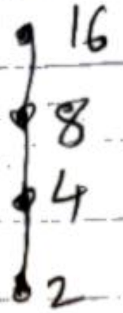


(*) ϕ is subset of every set so is placed at bottom

(*) $\{1\}$ & $\{2\}$ have no relation so kept at same level

(*) $\{1\} \subseteq \{1, 2\}$ & $\{2\} \subseteq \{1, 2\}$ so, $\{1, 2\}$ is placed above $\{1\}$ & $\{2\}$ & joined by straight line.

Ex1 Let $A = \{2, 4, 8, 16\}$. Then $(A, /)$ is a poset (infact chain)



Ex2 Let $A = \{1, 2, 3, 4\}$ and consider the relation
$R = \{(1,1), (1,2), (1,3), (2,2), (3,2), (3,3), (4,2), (4,3), (4,4)\}$.

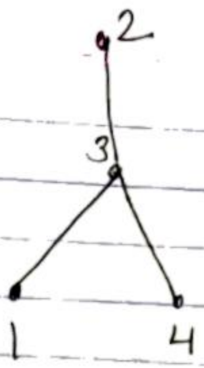
Prove R is partial ordering and draw Hasse diagram.

Soln → $(1,1), (2,2), (3,3), (4,4) \in R$ ∴ Reflexive.

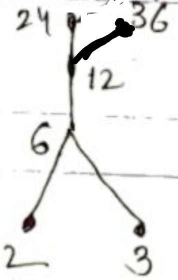
* (a,b) If aRb & $bRa \Rightarrow a = b$ then antisymmetric. This property also holds.

* $(1,2), (2,2) \in R \Rightarrow (1,2) \in R$
$(1,3), (3,3) \in R \Rightarrow (1,3) \in R$
$(4,2), (2,2) \in R \Rightarrow (4,2) \in R$ and similarly you can show other combinations. Hence transitive.

∴ R is a partial ordering

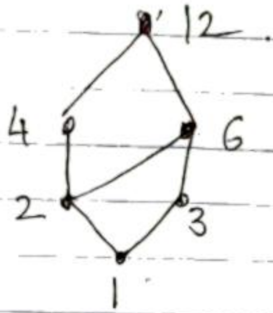


Ex3 $A = \{2, 3, 6, 12, 24, 36\}$, divisibility relation.
Draw Hasse diagram



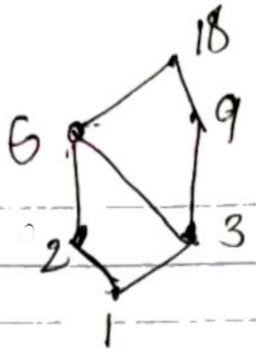
Ex4 Draw Hasse diagram of D_{12} (Divisors of 12) with divisibility relation.

$D_{12} = \{1, 2, 3, 4, 6, 12\}$.

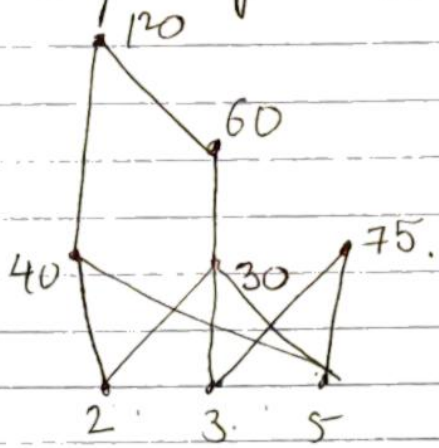


Ex5 Try $(D_{18}, /)$

$D_{18} = \{1, 2, 3, 6, 9, 18\}$.



Ex6. $S = \{2, 3, 5, 30, 40, 60, 75, 120\}$. Draw Hasse diagram of $(S, /)$.



Special Elements in Posets: Let $(S, \preceq)$ be a poset.

Maximal element: $a \in S$ is maximal element if $\nexists$ $x \in S$ s.t. $a \prec x$.

Minimal element: $b \in S$ is minimal element if $\nexists$ $x \in S$ s.t. $x \prec b$.

Greatest element: $a \in S$ is greatest element of S if $x \preceq a$ $\forall x \in S$. i.e, every element in S preceeds a. The greatest element, if it exists, is unique.

Least element: $b \in S$ is least element if $b \preceq x$ $\forall x \in P$. The least element if it exists is unique.

Note: Maximal & minimal elements may ~~ext~~ or may not exist. If they exist they may not be unique.

In Ex1 2 is least element as
$2 \preccurlyeq x \quad \forall x \in A = \{2, 4, 8, 16\}$
and 16 is the great element as $16 \succcurlyeq x \; \forall x \in A$

In Ex2
Maximal element = greatest element = 2
minimal elements are 1, 4.
No least element as 1 & 4 are not related

In Ex3 Maximal element = ~~greatest element~~ = 36, 24
minimal elts are 2, 3
No least element (why?)

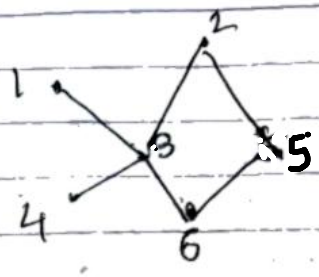
In Ex 4 maximal element = greatest element = 12
minimal element = least element = 1

In Ex 6 Maximal elts are 75, 120
No great element as 75 & 120 are not related.

2, 3, 5 are minimal elements
No least element.

Ex 7 Let $A = \{1, 2, 3, 4, 5, 6\}$ be ordered set shown in figure below. Find
(a) All minimal & maximal elements of A
(b) Greatest & least element of A

(c) All linearly ordered subsets of A, each of which contains atleast 3 ~~to~~ elements.



Sol(a) minimal elements are 4, 6.
maximal elements are 1, 2

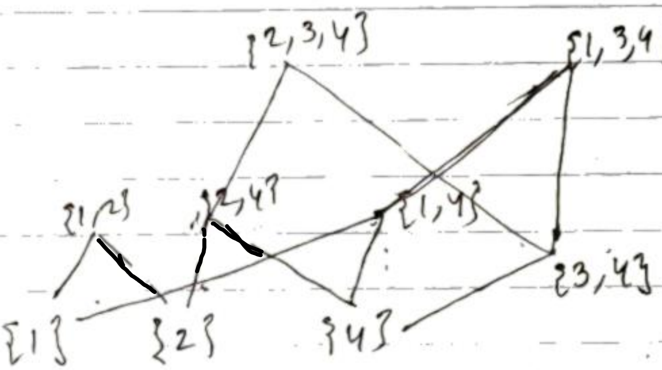
(b) Greatest element → DNE
least element → DNE

(c) {1, 3, 4}, {1, 3, 6}, {2, 3, 4}, {2, 3, 6}.
{2, 5, 6}

Ex Consider, Poset { {1}, {2}, {4}, {1, 2}, {1, 4}, {2, 4}, {3, 4}, {1, 3, 4}, {2, 3, 4}, ⊆ }.

Find (a) maximal elements (b) minimal elements
(c) all upper bounds of { {2}, {4} } & the lub, if it exists
(d) all lower bounds of {1, 3, 4} & glb, if it exists

Soln



(a) Maximal elements → {1,2}, {1,3,4}, {2,3,4}

(b) Minimal elements → {1}, {2}, {4}.

(c) Upper bounds of {{2},{4}} → {2,4}, {2,3,4}.

lub → {2,4}

(d) Lower bounds of {1,3,4} → {1}, {4}, {3,4}.

glb – {3,4}.

<u>Antichain</u>: A subset of a poset is called an antichain if every two elements of this subset are incomparable.

In <u>Ex 7</u> antichain with more than one elements are → {1,2}, {1,5}, {4,5} {3,5}.

In <u>Ex 8</u> Antichains with more than one element are → {{1,2}, {2,4}}, {{1,2}, {1,4}}

{{1,2}, {3,4}}, {{2,4}, {1,4}},

{{1,2}, {2,3,4}}, {{1,2}, {1,3,4}}, {{2,4}, {1,3,4}}

{{1,4}, {2,3,4}}, {{1,3,4}, {2,3,4}}