Representation of Relation: There are many ways of representations. For visualising of information, graphical methods are particularly useful and for mathematical calculations, matrix method is convienient.

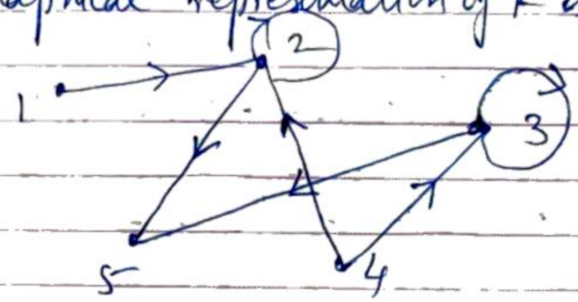
Graphs of Relations:

1. The elements of set on which relation is defined ~~are~~ are denoted by points called nodes
2. If xRy then draw a arc/line from x to y with arrow indicating the direction.
3. For xRx, we draw a loop starting & ending at x.

The graph in which directions of arcs are shown is called directed graph / digraph.

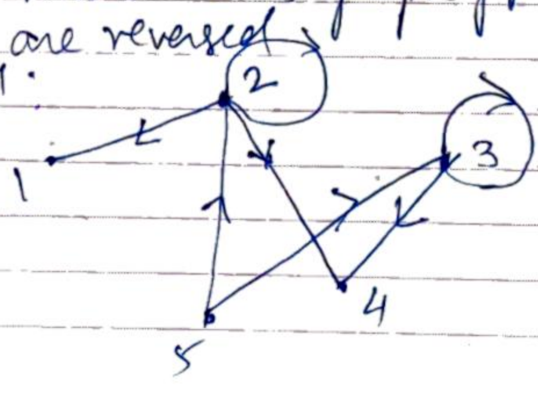
Ex Let $A = \{1, 2, 3, 4\}$ and $B = \{2, 3, 5\}$ and R be relation from A to B given by $R = \{(1,2), (2,5), (3,3), (3,5), (4,2), (4,3), (2,2)\}$.

Then graphical representation of R is



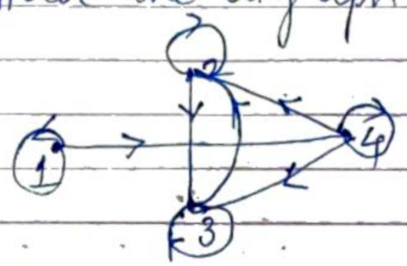
Note: (1) Elements of A & B can be put in any order
(2) Graph of R is same as graph of R^{-1} just the directions are reversed.

Graph of R^{-1}:



* If in a diagraph each node has a loop then relation is reflexive
* If to each edge/arc b/w two distinct vertices there is an edge/arc in the opposite direction then relation is ~~ref~~ symmetric
* If ~~to each a~~ Whenever there is a directed edge from a to b and from b to c then there is also a directed edge from a to c, then relation is transitive

Ex1 Consider the digraph

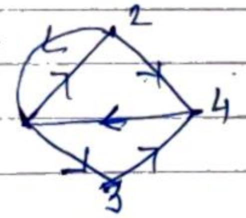


(1) Is the graph reflexive? Symmetric? transitive?
(2) Write the Relation.

Soln (1) Reflexive: Yes, loops at all the vertices/nodes
Symmetric: No. There is an edge from 4 to 2 but not from 2 to 4. same for 3 & 4 and 1 & 4
Transitive: No. There is an edge from 1 to 4 and 4 to 2 but no edge from 1 to 4
(Similarly you can find other)

(2) $R = \{(1,1), (2,2), (3,3), (4,4), (2,3), (3,2), (4,2), (4,3), (1,4)\}$.

Ex2



(1) Not reflexive, not symm & not transitive
(2) $R = \{(1,2), (2,1), (2,4), (4,1), (3,4), (1,3)\}$

Matrix Representation of Relation

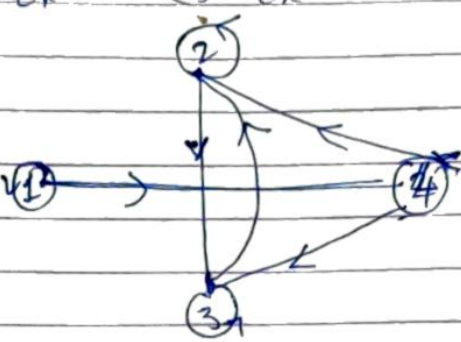
Let $A = \{a_1, a_2, \ldots a_m\}$ & $B = \{b_1, b_2, \ldots b_n\}$ and R be relation from A to B. Then matrix representation of R is

$$M_R = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R. \end{cases}$$

M_R is a $(m \times n)$ matrix (binary matrix)

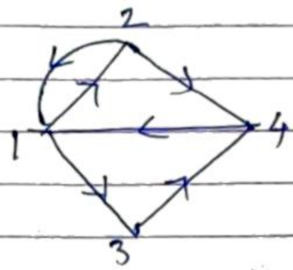
Ex Find matrix representation of graphs in
(1) Ex 1 (2) Ex 2.

Soln 1)



$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

(2)



$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

From matrix:

* If all diagonal entries are '1' then relation is reflexive
* If matrix is symmetric then relation symmetric
* If $m_{ij} = 1$ and $m_{jk} = 1 \Rightarrow m_{ik} = 1$ or, $M_R^2 + M_R = M_R$.

Ex Let $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ R is reflexive
R is symmetric

For transitive

$M_R^2 = M_R \cdot M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$M_R^2 + M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = M_R$.

$\therefore$ Transitive.

OR $m_{13}=1 \quad m_{31}=1 \Rightarrow m_{11}=1$ ✓ $\quad \therefore$ Transitive

$m_{31}=1 \quad m_{13}=1 \Rightarrow m_{33}=1$ ✓

$m_{11}=1 \quad m_{13}=1 \Rightarrow m_{13}=1$ ✓

$m_{31}=1 \quad m_{11}=1 \Rightarrow m_{31}=1$ ✓

<u>Theorem</u> If R is a relation on set A, then R is transitive iff $R^2 \subseteq R$.

<u>Pf</u>| Let R be transitive and $(x,z) \in R^2$.

$\Rightarrow \exists\, y \in A$ such that $(x,y) \in R$ & $(y,z) \in R$. (defn of R^2)

But R is transitive, so, $(x,z) \in R$.

$\Rightarrow R^2 \subseteq R$ //

<u>Conversely</u>| Let $R^2 \subseteq R$.

<u>T.P.T</u>| R is transitive.

Let (x,y) and $(y,z) \in R \Rightarrow (x,z) \in R^2$ (defn of R^2)

But $R^2 \subseteq R$ $\therefore (x,z) \in R$.

$\Rightarrow R$ is transitive //.

We had seen <u>composition of relations</u> was defined as,

$R \subseteq A \times B$ & $S \subseteq B \times C$ then $R \circ S \subseteq A \times C$

$R \circ S = \{(a,c) \in A \times C : \text{for some } b \in B \ (a,b) \in R \ \& \ (b,c) \in S\}$.

Thm Let A, B and C be finite sets. Let R be relation from A to B and S be relation from B to C. Show that $M_{RoS} = M_R \cdot M_S$, where M_R, M_S represent relation matrices of R and S respectively.

Proof: Let $A = \{a_1, a_2, \dots a_m\}$, $B = \{b_1, b_2, \dots b_n\}$ and $C = \{c_1, c_2, \dots c_p\}$.

Let $M_R = [a_{ij}]$ $M_S = [b_{ij}]$ and $M_{RoS} = [d_{ij}]$.

Then $d_{ij} = \begin{cases} 1 & (a_i, c_j) \in RoS \\ 0 & \text{otherwise} \end{cases}$.

Now, $(a_i, c_j) \in RoS \Rightarrow$ there is a b_k such that $(a_i, b_k) \in R$ and $(b_k, c_j) \in S$. i.e, $a_{ik} = 1$ & $b_{kj} = 1$, $1 \le k \le n$

$d_{ij} = 0 \Rightarrow$ either $(a_i, b_k) \notin R$ or $(b_k, c_j) \notin S$.

Now, if $a_{ik} = 1$ & $b_{kj} = 1$ then in $M_R \cdot M_S$ value at (i, j) place will be 1 which is same as $d_{ij} = 1$ if $(a_i, c_j) \in RoS$ and if $a_{ik} = 0$ or $b_{kj} = 0$ then in $M_R \cdot M_S$ value at (i, j) place will be 0 which is same as $d_{ij} = 0$ if $(a_i, c_j) \notin RoS$

Hence $M_{RoS} = M_R \cdot M_S$ //

Note: Composition of relations is distributive but not commutative (why ??).

Closure of Relations: A relation R may or may not be reflexive, symmetric or transitive. But by adding some pairs we can have the desired property. The smallest such relation on A is called closure of R w.r.t that property (ref/symm/transitive)

Closure of Relations using composition:

Reflexive closure:

$$R^{(r)} = R \cup I_A$$

where $I_A = \{(x,x) : x \in A\}$. (diagonal relation)

Symmetric closure: $R^{(s)} = R \cup R^{-1}$

Transitive closure: If $|A| = n$ then

$$R^{(T)} = R \cup R^2 \cup \cdots \cup R^n$$

Ex 3 Find reflexive, symmetric and transitive closure of R ~~By using composi~~ using composition of relations

(a) $R = \{(0,1), (1,2), (2,3)\}$ on $A = \{0,1,2,3\}$

Reflexive closure: $I_A = \{(0,0), (1,1), (2,2), (3,3)\}$

$\therefore R^{(r)} = R \cup I_A = \{(0,1), (1,2), (2,3), (0,0), (1,1), (2,2), (3,3)\}$.

Symmetric closure: $R^{-1} = \{(1,0), (2,1), (3,2)\}$

$\therefore R^{(s)} = R \cup R^{-1} = \{(0,1), (1,2), (2,3), (1,0), (2,1), (3,2)\}$

Transitive closure:

$R^2 = R \circ R = \{(0,2), (1,3)\}$

$R^3 = R^2 \circ R = \{(0,3)\}$. $R^4 = R^3 \circ R = \phi$.

$\therefore R^{(T)} = R \cup R^2 \cup R^3 = \{(0,1), (1,2), (2,3), (0,2), (1,3), (0,3)\}$

(b) $R = \{(a,b), (b,c), (c,c), (c,a), (c,b)\}$ on $A = \{a,b,c\}$.

Reflexive closure $= R \cup I_A = \{(a,b), (b,c), (c,c), (c,a), (c,b), (a,a), (b,b)\}$

Symmetric closure $= R \cup R^{-1} = \{(a,b), (b,c), (c,c), (c,a), (c,b), (b,a), (a,c)\}$.

Transitive closure:

$R^2 = R \circ R = \{(a,c), (b,c), (b,a), (b,b), (c,a), (c,b), (c,c)\}$

$R^3 = R^2 \circ R = \{(a,c), (a,a), (a,b), (b,c), (b,a), (b,b), (c,b)$
$(c,c), (c,a)\}$ eq

$R^{(T)} = R \cup R^2 \cup R^3 = \{(a,b), (b,c), (c,c), (c,a), (c,b), (a,c)$
$(b,a), (b,b), (a,a)\}$.

Closure of Relations using matrices:

Reflexive: $R^{(r)} = R \quad M_R^{(r)} = M_R \vee I_n$.

Symmetric closure: $M_R^{(s)} = M_R \vee M_R^T$

Transitive closure: $M_R^{(T)} = M_R \vee M_R^2 \vee \cdots M_R^n$

Ex Find reflexive, symmetric and transitive closure of R using matrix method of relations in Ex3.

(a) $R = \{(0,1), (1,2), (2,3)\}$ on $A = \{0,1,2,3\}$

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_R^{(r)} = M_R \vee I_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Reflexive closure

$$= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\therefore R^{(r)} = \{(0,0), (0,1), (1,1), (1,2), (2,2), (2,3), (3,3)\}$

$$M_R^{(s)} = M_R \vee M_R^T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Symmetric closure

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\therefore R^{(s)} = \{(0,1), (1,0), (1,2), (2,1), (2,3), (3,2)\}$

$M_R^T = M_R \vee M_{R^2} \vee M_{R^3} \vee M_{R^4}$.

$$M_{R^2} = M_R \cdot M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Transitive closure

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{R^3} = M_{R^2} \cdot M_R = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{R^4} = M_{R^3} \cdot M_R = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore M_R^{(T)} = \begin{bmatrix} 0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0 \end{bmatrix} \vee \begin{bmatrix} 0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{bmatrix} \vee \begin{bmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} \vee \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}$

$= \begin{bmatrix} 0&1&1&1\\0&0&1&1\\0&0&0&1\\0&0&0&0 \end{bmatrix}$

$R^{(T)} = \{(0,1),(0,2),(0,3),(1,2),(1,3),(2,3)\}$.

(b) $R = \{(a,b),(b,c),(c,c),(c,a),(c,b)\}$ on $A = \{a,b,c\}$

$M_R = \begin{bmatrix} 0&1&0\\0&0&1\\1&1&1 \end{bmatrix}$

$M_R^{(r)} = \begin{bmatrix} 0&1&0\\0&0&1\\1&1&1 \end{bmatrix} \vee \begin{bmatrix} 1&0&0\\0&1&0\\0&0&1 \end{bmatrix} = \begin{bmatrix} 1&1&0\\0&1&1\\1&1&1 \end{bmatrix}$

Reflexive closure

$R^{(r)} = \{(a,a),(a,b),(b,b),(b,c),(c,a),(c,b),(c,c)\}$

$M_R^{(S)} = \begin{bmatrix} 0&1&0\\0&0&1\\1&1&1 \end{bmatrix} \vee \begin{bmatrix} 0&0&1\\1&0&1\\0&1&1 \end{bmatrix} = \begin{bmatrix} 0&1&1\\1&0&1\\1&1&1 \end{bmatrix}$

Symmetric closure

$R^{(S)} = \{(a,b),(a,c),(b,a),(b,c),(c,a),(c,b),(c,c)\}$

$M_R^{(T)} = M_R \vee M_{R^2} \vee M_{R^3}$

$M_{R^2} = \begin{bmatrix} 0&1&0\\0&0&1\\1&1&1 \end{bmatrix} \begin{bmatrix} 0&1&0\\0&0&1\\1&1&1 \end{bmatrix} = \begin{bmatrix} 0&0&1\\1&1&1\\1&1&1 \end{bmatrix}$

Transitive closure

$M_{R^3} = \begin{bmatrix} 0&0&1\\1&1&1\\1&1&1 \end{bmatrix} \begin{bmatrix} 0&1&0\\0&0&1\\1&1&1 \end{bmatrix} = \begin{bmatrix} 1&1&1\\1&1&1\\1&1&1 \end{bmatrix}$

$$\therefore M_R^{(T)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$R^{(T)} = \{(a,a), (a,b), (a,c), (b,a), (b,b), (b,c), (c,a), (c,b), (c,c)\}$ //

Warshall's Algorithm (for finding transitive closure).

Let $A = \{a_{11}, a_{21}, \ldots a_{n1}\}$ be a non-empty set and R be relation on A.

~~1) Let $W_0 = M_R$~~

~~2) $W_K = \begin{cases} 1 & \text{if there is an edge from } v_i \text{ to } v_j \text{ or there is a path (of length 2) from } v_i \text{ to } v_l \text{ and } v_l \text{ to } v_j \\ 0 & \text{otherwise} \end{cases}$~~

1) Let $W_0 = M_R$
2) Transfer all 1's of W_0 to W_1 (keeping the places intact)
3) Consider 1st column & 1st row of W_1. List the locations $p_1, p_2, \ldots p_r$, $1 \le r \le n$ in column 1 of W_0 where entry is 1 and locations $q_1, q_2, \ldots q_k$, $1 \le k \le n$ of row 1 of W_1 where entry is 1
4) place 1 at locations (p_i, q_j) if 1 is already not there
5) Hence we get W_1.
6) Repeat steps from 2–4 with 2^{nd} column & 2^{nd} row to get W_2.
7) Repeat till all rows and colums are covered

Ex Find transitive closure of $R = \{(0,1), (1,2), (2,3)\}$ on $A = \{0,1,2,3\}$ using Warshall's algorithm

soln

$$W_0 = M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1st column → no 1.
1st row → 2
∴ No change i.e, $W_1 = W_0$.

2nd column → 1.
2nd row → 3 ∴ Place 1 at (1,3) i.e, $w_{13} = 1$

$$W_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

3rd column → 1, 2 Place 1 at (1,4), (2,4). i.e, $w_{14} = 1$
3rd row → 4. ∴ $w_{24} = 1$

$$\therefore W_3 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

4rth column → 1, 2, 3. ∴ No change i.e, $W_4 = W_3$
4rth row → no 1.

$\therefore R^{(T)} = \{(0,1), (0,2), (0,3), (1,2), (1,3), (2,3)\}$

(h) $R = \{(a,b), (b,c), (c,c), (c,a), (c,b)\}$ on $A = \{a,b,c\}$

$$W_0 = M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

1st column → 3 1st row → 2 ∴ (3,2) place → 1 which already has 1 ∴ $W_1 = W_0$.

2nd column → 1, 3 2nd row → 3.

∴ Place 1 at (1,3), (3,3) → This is already one

$$W_2 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

3^{rd} column $\rightarrow$ 1, 2, 3

3^{rd} row $\rightarrow$ 1, 2, 3

$\therefore$ place 1 at (1,1), (1,2), (1,3), (2,1), (2,2), (2,3)

(3,1), (3,2), (3,3) $\rightarrow$ These already haves

$$\therefore W_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} //$$

(c) {(1,2), (2,1), (2,3), (3,4)} on A = {1,2,3,4}

Ans $R^{(T)}$ = {(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,4)} //