Bessel's Inequality →

Statement → For every square Integrable function $f(x)$

$$\sum_{i=1}^{\infty} |C_i|^2 = \sum_{i=1}^{\infty} |(f, \varphi_i)|^2 \leq \|f\|^2$$

where $f(x)$ is real & Continuous and $\varphi_i(x)$: $i = 1, 2$ - - - is real & Continuous and consisting normalized orthogonal set

Proof → Consider

$$\int_a^b |f(x) - \sum C_i \varphi_i(x)|^2 dx = \int_a^b |f(x)|^2 dx + \sum_{i=1}^{n} \int_a^b |C_i|^2 |\varphi_i(x)|^2 dx$$

$\boxed{(a+b)^2 = a^2 + b^2 - ab - ab}$

$$- \sum_{i=1}^{n} \int_a^b \bar{f}(x) C_i \varphi_i(x) dx - \sum_{i=1}^{n} \int_a^b f(x) \bar{C_i} \bar{\varphi_i}(x) dx \quad \text{①}$$

Now Since $\int_a^b |\varphi_i(x)|^2 dx = 1$ given $\varphi(x)$ is real & continuous and consisting normalized orthogonal set

$$\int_a^b f(x) \bar{\varphi}_i(x) dx = C_i$$

$$\& \int_a^b \bar{f}(x) \varphi_i(x) dx = \bar{C_i}$$

$$\int_a^b |f(x) - \sum_{i=1}^{n} C_i \varphi_i(x)|^2 dx = \int_a^b |f(x)|^2 dx + \cancel{\sum_{i=1}^{n} |C_i|^2} dx$$

$$\cancel{- \sum_{i=1}^{n} C_i \bar{C_i}} - \cancel{\sum_{i=1}^{n} C_i \bar{C_i}} = \cancel{- \sum_{i=1}^{n} |C_i|^2} - \sum_{i=1}^{n} |C_i|^2 \quad \text{②}$$

$$\int_a^b |f(x) - \sum_{i=1}^{n} c_i \varphi_i(x)|^2 \, dx = \int_a^b |f(x)|^2 \, dx - \sum_{i=1}^{n} |c_i|^2$$

The integral in (2) has non-negative

therefore, for every n, we have

$$\sum_{i=1}^{n} |c_i|^2 \le \int_a^b |f(x)|^2 \, dx = \|f\|^2$$

when $n \to \infty$ the R.H.S of (2) tends to zero

See Down

It follows that $\sum_{i=1}^{n} |c_i|^2$ is always convergent and its sum satisfies the inequality

$$\sum_{i=1}^{n} |c_i|^2 \le \int_a^b |f(x)|^2 \, dx = \|f\|^2$$

Proved.