RIESZ - FISCHER THEOREM →

Let $\{\varphi_i(x)\}$ is an orthogonal system of L_2 - functions defined and integrable together with the squares of their moduli in the domain of (a,b) and $\langle \alpha_i \rangle$ is a sequence of complex numbers, then the series

$$\sum_{i=1}^{\infty} |\alpha_i|^2$$

Converges. There exists a unique function $f(x)$, integrable together with the square of its modulus for which the numbers α_i are the Fourier Coefficients with regard to an orthogonal system $\{\varphi_i(x)\}$ to which the Fourier series Converges in the mean.

Proof → Consider the sequence of partial sums

$$S_n(x) = \sum_{i=1}^{n} \alpha_i \varphi_i(x) \rightarrow (1)$$

Since We Know that

$$\int_a^b |S_{n+m}(x) - S_n(x)|^2 dx = |\alpha_{n+1}|^2 + |\alpha_{n+2}|^2 + \cdots + |\alpha_{n+m}|^2 \rightarrow (2)$$

$$|\alpha_1\varphi_1(x) + \alpha_2\varphi_2(x) - \cdots \alpha_n\varphi_n(x) + \alpha_{n+1}\varphi_{n+1}(x) - \cdots \alpha_{n+m}\varphi_{n+m}(x) - \{\alpha_1\varphi_1(x) + \alpha_2\varphi_2(x) - \cdots \alpha_n\varphi_n(x)\}|$$

Since ~~that~~ the sequence $\sum_{i=1}^{\infty} |\alpha_i|^2$ is given in Theorem Convergent for every $\epsilon > 0$, there exist an arbitrary chosen positive small number δ such that

$$\| S_{n+m}(x) - S_n(x) \| < \epsilon \quad \forall\, n > \delta, m \rightarrow (3)$$

Thus, there exists a unique function $f(x)$ (giving integrable in the domain (a, b) together with the square of its modulus to which the sequence (1) converges in the mean & therefore

$$\| f(x) - \sum_{i=1}^{n} \alpha_i \varphi_i(x) \| \rightarrow 0 \quad \text{as } n \rightarrow \infty \rightarrow (4)$$

Since the numbers α_i, are the Fourier Coefficients of the function $f(x)$ with regard to the system $\{\varphi_i(x)\}$ then

$$\| f - \sum_{i=1}^{n} \alpha_i \varphi_i \|^2 = \| f \|^2 + \sum_{i=1}^{n} |c_i|^2 + \sum_{i=1}^{n} |\alpha_i - c_i|^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

$\rightarrow (5)$

Using Bessel's Inequality

$$\alpha_i = c_i = \int_a^b f(x) \overline{\varphi_i(x)}\, dx \rightarrow (6)$$

The Fourier Series $\sum_{i=1}^{\infty} c_i \varphi_i(x)$ of the function $f(x)$ with regard to the system $\{\varphi_i(x)\}$ is convergent in the mean to that function that is

$$\| f - \sum_{i=1}^{n} c_i \varphi_i \| \rightarrow 0 \quad \text{as } n \rightarrow \infty \rightarrow (7)$$

(5) We observed that the Fourier Coefficients c_i with the system $\{\varphi_i(x)\}$ satisfy Parseval's Eq