...symmetric kernel with a norm not zero has at least one Eigen Value

or

state and prove The HILBERT'S THEOREM

Proof → The recursive relation for iterated Kernel are

$$K_{l+m} = \int_a^b K_l(x,z)\, K_m(z,t)\, dz \to (1)$$

so $$K_{2n} = \int_a^b K_n(x,z)\, K_n(z,t)\, dz \to (*)$$

Consider the Fredholm Integral equation

$$u(x) = F(x) + \lambda \int_a^b K(x,t)\, u(t)\, dt \to (2)$$

The Solution is terms of a power series in λ

$$u(x) = F(x) + \sum_{n=1}^{\infty} \lambda^n \int_a^b K_n(x,t)\, F(t)\, dt \to (3)$$

Multiplying (3) by $\bar{F}(x)$ and Integrate in an Interval $[a,b]$

$$\int_a^b u(x)\bar{F}(x)\, dx \overset{= \int_a^b F(x)\bar{F}(x)dx}{=} + \sum_{m=1}^{\infty} \lambda^n \int_a^b \int_a^b \underbrace{K_n(x,t)\bar{F}(x)\, F(t)\, dx\, dt}_{U_n}$$

$$\int_a^b u(x)\,\bar{F}(x)dx = \int_a^b F(x)\bar{F}(x)\,dx + \sum_{n=1}^{\infty} \lambda^n U_n$$

$$U_n = \int_a^b \int_a^b K_n(x,t)\, \bar{F}(x)\, F(t)\, dx\, dt$$

so $$\int_a^b u(x)\bar{F}(x)\, dx = \sum_{n=0}^{\infty} U_n \lambda^n \to (4)$$

with $U_0 = \int_a^b F(x)\bar{F}(x)\, dx$

with $K_0(x,t) = 1$, $K_1(x,t) = K(x,t) \to (5)$

due to symmetry of the Kernel

$$K_n(z,t) = \bar{K}_n(t,z)$$

By (*) Eq

$$K_{2n} = \int_a^b K_n(x,z)\, K_n(z,t)\, dz$$

$U_n = \int_a^b \int_a^b K_n(x,t)$

so

$U_{2n} = \int_a^b \int_a^b K_{2n}(x,t)\, \bar{F}(x) F(t)\, dx\, dt$

$$\text{so } U_{2n} = \int_a^b \left[\int_a^b K_n(x,z)\, \bar{F}(x)\, dx\right] \cdot \left[\int_a^b K_n(z,t)\, F(t)\, dt\right] dz$$

$\because K_n(z,t) = \bar{K}_n(t,z)$ By symmetric

$$U_{2n} = \int_a^b \left|\int_a^b K_n(x,z)\, \bar{F}(x)\, dx\right|^2 dz \quad \text{as } z\bar{z} = |z|^2 \rightarrow (6)$$

It follows that all coefficients of series (6) with even subscripts are non-negative real numb

$$U_{2n} \geq 0$$

Also from (1) & (5) we have

$$U_{2n} = \int_a^b \left[\left|\int_a^b K_{n-1}(x,z)\, \bar{F}(x)\, dx\right| \left|\int_a^b \bar{K}_{n+1}(t,z)\, F(t)\, dt\right|\right] dz$$

By Schwarz Inequality

$$U_{2n}^2 \leq \left[\int_a^b \left|\int_a^b K_{n-1}(x,z)\, \bar{F}(x)\, dx\right|^2 dz\right] \left[\int_a^b \left|\int_a^b K_{n+1}(x,z)\, \bar{F}(x)\right|^2 dz\right]$$

By (6) we may written as

$$U_{2n}^2 \leq U_{2n-2}\, U_{2n+2} \quad \forall\, n \geq 2$$

Divide Both Sides by U_{2n-2}

$$\frac{U_{2n+2}}{U_{2n}} \geq \frac{U_{2n}}{U_{2n-2}}$$

Putting the Values of $n = 2, 3 \cdots n$ successively

$$\frac{U_{2n+2}}{U_{2n}} \geq \frac{U_{2n}}{U_{2n-2}} \geq \cdots \geq \frac{U_6}{U_4} \geq \frac{U_4}{U_2}$$

Therefore $\dfrac{U_{2n+2}}{U_{2n}} \geq \dfrac{U_4}{U_2}$

or $$\frac{U_{2n+2}}{U_{2n}} \frac{\lambda^{2n+2}}{\lambda^{2n}} \geq \frac{U_4 \lambda^4}{U_2 \lambda^2}$$

Therefore the Series diverge if

$$\frac{U_4}{U_2} |\lambda|^2 \geq 1$$

$$|\lambda| \geq \sqrt{\frac{U_2}{U_4}}$$

Hence one eigen Value of the kernel $K(x,t)$ is in the interval $-\sqrt{\frac{U_2}{U_4}}, +\sqrt{\frac{U_2}{U_4}}$ which is real.

Proved