Hilbert - Schmidt Theory

If a function is Riemann Integrable then it is also a Lebegue integrable.

A function $f(x)$ is said to be square integrable if

$$\int_a^b |f(x)|^2\, dx < \infty$$

A square integrable function $f(x)$ is also Called an L_2 function or I_2 function.

Now the Kernel $K(x,t)$ is regular and hence $K(x,t)$ is an L_2- function if the following Conditions are satisfied

i) $\displaystyle\int_a^b \int_a^b |K(x,t)|^2\, dx\, dt < \infty \quad a \le x \le b,\ a \le t \le b$

(ii) $\displaystyle\int_a^b |K(x,t)|^2\, dt < \infty \quad a \le x \le b$

(iii) $\displaystyle\int_a^b |K(x,t)|^2\, dt < \infty \quad a \le t \le b$

Scaler product → The scaler product of two function $f(x)$ and $g(x)$ is denoted by symbol (f,g) and defined as

$$(f,g) = \int_a^b f(x)\, \bar{g}(x)\, dx$$ where bar denotes the complex conjugate

like if $z = x+iy$ then $\bar{z} = x-iy$

Norm of a function → The norm of a complex function is the positive square root of the scaler product of the function by itself

$$\|f\| = \sqrt{(f,f)} = \left[\int_a^b f(x)\, \bar{f}(x)\, dx\right]^{1/2} = \left[\int_a^b |f(x)|^2 dx\right]^{1/2}$$

$\because z\bar{z} = |z|^2$

Orthogonal function → Two functions $f(x)$ and $g(x)$ is said to be orthogonal if $(f,g) = 0$ i.e

$$\int_a^b f(x)\, \bar{g}(x)\, dx = 0$$

Normalized function → A function $f(x)$ is said to be normalized if $\|f(x)\| = 1$