Normal Distribution: This is a continuous probability distribution. Another name for this is Gaussian distribution.

~~Denif~~ Density fn $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$ $\mu \to$ mean, $\sigma^2 \to$ variance.

$-\infty < x < \infty$

Corresponding distribution fn

$$F(x) = P(X \le x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-(x-\mu)^2/2\sigma^2} dx$$

Let Z be standardized variable corresponding to X i.e, $Z = \frac{X-\mu}{\sigma}$, then $E(Z) = \frac{1}{\sigma} E(X-\mu) = 0$

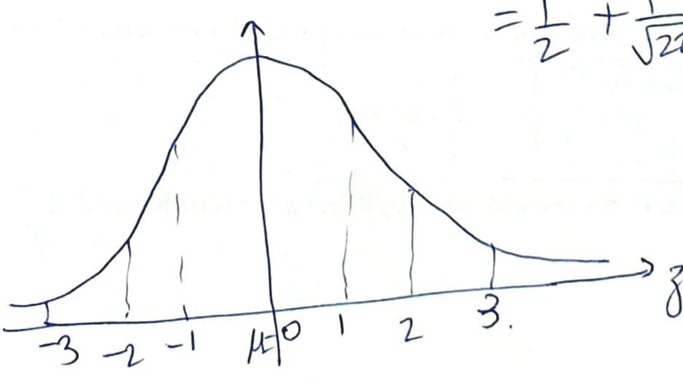
& $Var(Z) = \frac{1}{\sigma^2} E(X-\mu)^2 = \frac{1}{\sigma^2} Var(X)$

$= \frac{\sigma^2}{\sigma^2} = 1.$

and density fn $f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}.$

distribution fn $F(z) = P(Z \le z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2} du$

$= \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_{0}^{z} e^{-u^2/2} du$ //.



$P(-1 \le z \le 1) = 68.27\%$.

$P(-2 \le z \le 2) = 95.45\%$

$P(-3 \le z \le 3) = 99.75\%$.

$\cong P(\mu-\sigma \le X \le \mu+\sigma) = 68.27\%$.

i.e, area within s.d one of mean is 68.67%.

<u>Relation between Binomial & Normal:</u> If n is large and neither p nor q is too close to zero then binomial can be approximated by normal distribution with standardized variable $Z = \frac{X-\mu}{\sigma}$.

* Approximation becomes better as n increases and is exact in the limiting case.

In practice, np & $nq \ge 5$.

$$\lim_{n\to\infty} P\left(a \le \frac{X-np}{\sqrt{npq}} \le b\right) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2}\,dx$$

<u>Relation between Poisson and Normal:</u>

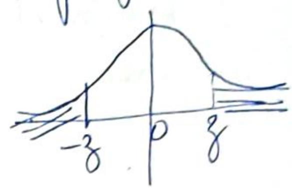
$$\lim_{\lambda\to\infty} P\left(a \le \frac{X-\lambda}{\sqrt{\lambda}} \le b\right) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-u^2/2}\,du.$$

Prop 1: $F(-z) = 1 - F(z)$

Pf: $F(-z) = P(Z \le -z) = P(Z \ge z)$ (by symmetry)

$= 1 - P(Z \le z)$

$= 1 - F(z)$ //



Prop 2: $P(a \le X \le b) = P(z_a \le Z \le z_b)$ where $z_a = \frac{a-\mu}{\sigma}$

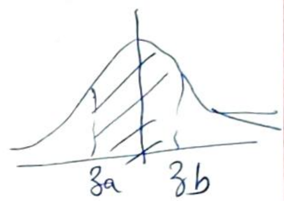
$= F(z_b) - F(z_a)$

Pf: $P(a \le X \le b) = P\left(\frac{a-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{b-\mu}{\sigma}\right)$

$= P(z_a \le Z \le z_b)$

$= P(Z \le z_b) - P(Z \le z_a)$

$= F(z_b) - F(z_a)$



Moment generating function:

$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$ $\quad (\because M(t) = E(e^{tx}))$

$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-(x-\mu)^2/2\sigma^2} dx$

Let $\frac{x-\mu}{\sigma} = z \quad dz = \frac{dx}{\sigma}$

$\Rightarrow x = \mu + \sigma z$.

$\therefore M_X(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\mu+\sigma z)} e^{-z^2/2} dz$.

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2\sigma t z)} dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}[(z-\sigma t)^2 - \sigma^2 t^2]} dz.$$

$$= \frac{e^{\mu t + (\sigma^2 t^2/2)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-\sigma t)^2} dz$$

Let $z - \sigma t = u \quad dz = du$.

$$= e^{\mu t + (\sigma^2 t^2/2)} \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2} du}_{=1} = e^{\mu t + \left(\frac{\sigma^2 t^2}{2}\right)} \quad //$$

<u>Note</u>: $M_Z(t) = e^{t^2/2}$ where ~~X=~~ $Z = \frac{X-\mu}{\sigma}$

<u>Odd Moments about mean:</u>

$$\mu_{2n+1} = \int_{-\infty}^{\infty} (x-\mu)^{2n+1} f(x) dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^{2n+1} e^{-(x-\mu)^2/2\sigma^2} dx.$$

Let $z = \frac{x-\mu}{\sigma} \quad \therefore dz = \frac{dx}{\sigma}$

$$\therefore \mu_{2n+1} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z)^{2n+1} e^{-z^2/2} dz.$$

$$= \frac{\sigma^{2n+1}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{2n+1} e^{-z^2/2} dz = 0.$$

$\left(\int_{-a}^{a} f(x) dx = 0\right.$ for odd fn $f(x)$).

Even Moments about mean:

$$\mu_{2n} = \int_{-\infty}^{\infty} (x-\mu)^{2n} f(x)\,dx$$

$$= \frac{\sigma^{2n}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{2n} e^{-z^2/2}\,dz \qquad \left(\frac{x-\mu}{\sigma} = z \right)$$

Put $z^2/2 = t$ $\quad z\,dz = dt$ or, $dz = \frac{dt}{\sqrt{2t}}$.

$$= \frac{\sigma^{2n}}{\sqrt{2\pi}} \cdot 2 \int_{0}^{\infty} \cancel{z^{2n}} (2t)^{n} e^{-t} \frac{dt}{\sqrt{2t}} \qquad (\text{For even fn}$$

$$= \frac{2^n \sigma^{2n}}{\sqrt{\pi}} \int_0^{\infty} t^{n-\frac{1}{2}} e^{-t}\,dt$$

$$= \frac{2^n \sigma^{2n}}{\sqrt{\pi}} \int_0^{\infty} e^{-t}\, t^{(n+\frac{1}{2})-1}\,dt = \frac{2^n \sigma^{2n}}{\sqrt{\pi}} \Gamma\left(n+\tfrac{1}{2}\right) \,//$$

$$\left(\Gamma n = \int_0^{\infty} e^{-t} t^{n-1}\,dt \right.$$

$$\Gamma n = (n-1)\,\Gamma(n-1)$$

Note: $P(a \le X \le b) = \int_a^b f(x)\,dx$. (for continuous r.v. X)

Ex If $Z \sim N(0,1)$ find $P(0 \le Z \le 1.40)$

soln $P(0 \le Z \le 1.40) = F(1.40) - F(0)$

$$= 0.9192 - 0.5 = 0.4192$$

Ex If $Z \sim N(0,1)$ find

(a) $P(-2.00 \le Z \le 0)$ (b) $P(-1.09 \le Z \le -0.06)$

(c) $P(|Z| \le 1)$

Sol (a) $P(-2.00 \le Z \le 0) = \cancel{F(0) - F(-2.00)}$

$= P(0 \le Z \le 2.00)$ $\cancel{= F(0) - [1 - F(2.00)]}$ $\cancel{= 0.5 - [1 - 0.9772]}$

$= F(2) - F(0)$

$= 0.9772 - 0.5$

$= 0.4772$

(b) $P(-1.09 \le Z \le -0.06) = P(0.06 \le z \le 1.09)$

$= F(1.09) - F(0.06) = 0.8621 - 0.5239$

$= 0.3382$

(c) $P(|z| \le 1) = P(-1 \le z \le 1) = 2P(0 \le z \le 1)$

$= 2[F(1) - F(0)] = 2[0.8413 - 0.5]$

$= 0.6826$

Ex If $X \sim N(4, 25)$ find

(a) $P(X > 19)$ (b) $P(4 < X < 10)$

Sol $P(X > 19) = P\left(Z > \frac{19-4}{5}\right)$ (here $\mu = 4$, $\sigma^2 = 25 \therefore \sigma = 5$)

$= P(Z > 3)$

$= 1 - F(3)$

$= 1 - 0.9987 = 0.0013$ //

(b) $P(4 < X < 10) = P\left(0 < Z < \frac{6}{5}\right)$

$= F(1.2) - F(0)$

$= 0.8849 - 0.5.$

$= 0.3849.$

Ex If mgf of r.v. X is $M(t) = \exp(166t + 200t^2)$

find (a) μ (b) σ^2 (c) $P(170 < X < 200)$

Soln $M(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$ for $X \sim N(\mu, \sigma^2)$

(a) $\mu = 166$ (b) $\frac{\sigma^2}{2} = 200 \Rightarrow \sigma^2 = 400.$

(c) $P(170 < X < 200) = P\left(\frac{1}{5} < Z < 1.7\right)$ $\left(Z = \frac{X-\mu}{\sigma}\right)$

$= F(1.7) - F(0.2)$

$= 0.9554 - 0.5793 = 0.3761$ //