INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS AND SHOCK

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OUTLINES

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Partial differential equations arise in geometry and physics when the number of independent variables in the problem under discussion is two or more. For instance, in the study of thermal effects in solid body the temperature θ may varies from point to point in the solid as well as from time to time, and , as a consequence, the derivatives

$$\frac{\partial \theta}{\partial x}, \qquad \frac{\partial \theta}{\partial y}, \qquad \frac{\partial \theta}{\partial z}, \qquad \frac{\partial \theta}{\partial t}$$

will, in general, be non zero. Furthermore in any particular problem it may happen that higher derivatives of the types

$$\frac{\partial^2 \theta}{\partial x^2}, \qquad \frac{\partial^2 \theta}{\partial x\, \partial t}, \qquad \frac{\partial^3 \theta}{\partial x^2\, \partial t}, \quad \text{etc.}$$

may be of physical significance.

When the laws of physics are applied to a problem of this kind , we some times obtain a relation between the derivatives of the kind

$$F\left(\frac{\partial\theta}{\partial x},\ldots\ldots,\frac{\partial^2\theta}{\partial x\,\partial t},\ldots\ldots,\frac{\partial^2\theta}{\partial x\,\partial t},\ldots\right)=0 \qquad (1)$$

Such an equation relating partial derivatives is called a partial differential Equation.

Origins of First-order PDEs:

we consider the equation

$$x^2+y^2+(z-c)^2=a^2 \qquad (2)$$

in which the constants a and c are arbitrary. The equation (2) represents the set of all spheres whose centers lie along the z axis.

we differentiate this equation with respect to x and y and eliminating c we have

$$yp - xq = 0 \tag{3}$$

where $$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}$$

Equation (3) is of first order. In some sense, then, the set of all spheres with centers on the z axis is characterized by PDE (3). However, other geometrical entities can be described by the same equation. For example, the equation

$$x^2 + y^2 = (z - c)^2 \tan^2 \alpha \tag{4}$$

In which both of the constants c and α are arbitrary , represents the set of

Right circular cones whose axes are coincide with the line oz.

Now what the sphere and cones have in common is that they are surfaces of revolution which have the line oz as axes of symmetry. All surfaces of revolution with this property are characterized by an equation of the form

$$z = f\left(x^2 + y^2\right) \qquad (5)$$

where the function f is arbitrary.

Thus we see that the function z defined by each of the equations (2), (4) and (5) is in some sense, a "solution" of the equation (3).

The relations (2) and (4) are both of type

$$F(x, y, a, b) = 0 \qquad (6)$$

now differentiate this equation with respect to x and y and eliminating a and b we get

$$F(x, y, p, q) = 0 \qquad (7)$$

The obvious generalization of the relation (5) relation between x, y and z of the type

$$F(u, v) = 0 \qquad (8)$$

where u and v are known functions of x, y and z and F is an arbitrary function of u and v.

If we differentiate equation (8) with respect to x and y, respectively, we obtain equations

$$\frac{\partial F}{\partial u}\left\{\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right\}+\frac{\partial F}{\partial v}\left\{\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right\}=0$$

$$\frac{\partial F}{\partial u}\left\{\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right\}+\frac{\partial F}{\partial v}\left\{\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right\}=0$$

and if we eliminate $\partial F/\partial u$ and $\partial F/\partial v$ from these equations, we obtain

$$p\frac{\partial(u,v)}{\partial(y,z)}+q\frac{\partial(u,v)}{\partial(z,x)}=\frac{\partial(u,v)}{\partial(x,y)} \qquad (9)$$

which is a partial differential equation of type (7).

it should be observed, however, that the PDE (9) is linear equation whereas equation (7) need not be linear. For example, the equation

$$(x-a)^2+(y-b)^2+z^2=1$$

leads to the first - order nonlinear differential equation

$$z^2\left(1+p^2+q^2\right)=1$$

First Order Quasi Linear PDEs

The general quasilinear system of n first-order PDEs in n functions of two independent variables is

$$\sum_{j=1}^{n} a_{ij} \frac{\partial u_j}{\partial x} + \sum_{j=1}^{n} b_{ij} \frac{\partial u_j}{\partial y} = c_{ij} \qquad \text{i = (1,2,3,….)} \qquad (1)$$

where a_{ij} , b_{ij} and c_{ij} may depend on x, y, $u_1, u_2, \ldots u_n$ If each a_{ij} and b_{ij} is independent of $u_1, u_2, \ldots u_n$ the system (1) is called almost linear. If, in addition, each c_{ij} depends linearly on $u_1, u_2, \ldots u_n$ the system is said to be linear.

In terms of the n x n matrices $A = \left[a_{ij} \right]$ and $B = \left[b_{ij} \right]$ and the column vectors $u = \left(u_1, u_2, ..., u_n \right)^T$ and $c = \left(c_1, c_2, ..., c_n \right)^T$, the system (1) can be expressed as

$$Au_x + Bu_y = c \qquad (2)$$

If A or B is nonsingular, it is usually possible to classify system (2) according to type. Suppose $\det B \neq 0$ and define a polynomial of degree n in λ by

$$P_n(\lambda) \equiv \det\left(A^T - \lambda B^T \right) = \det\left(A - \lambda B \right) \qquad (3)$$

System (2) is classified as

Elliptic if $P_n(\lambda)$ has no real zeros.

Hyperbolic if $P_n(\lambda)$ has n real, distinct zeros; or if $P_n(\lambda)$ has n real zeros, at least one of which is repeated, and the generalized eigenvalues problem $\left(A^T - \lambda B^T\right)t = 0$ yields n linearly independent eigenvectors t .

Parabolic if $P_n(\lambda)$ has n real zeros, at least one of which is repeated, and the above generalized eigenvalue problem yields fewer than n linearly independent eigenvectors.

Nonlinear Hyperbolic Waves

Hyperbolic waves having nonlinearity of a special type i.e., waves governed by quasilinear hyperbolic partial differential equations.

The simplest example of a linear hyperbolic partial differential equation in two independent variables x and t is

$u_t + c u_x = 0$ c = real constant $(x, t) \in R^2$ (1)

Its solution $u = u_0\left(x - ct\right)$, where $u_0 : R \to R$ is an arbitrary real function with continuous first derivatives, represents a wave. Every point of its profile propagates with the same constant velocity c.

Consider now a nonlinear equation

$$u_t + u u_x = 0 \qquad (2)$$

whose solutions represent waves in which the velocity of propagation of a point on the pulse is equal to the amplitude at that point. This equation is called Burgers' equation.

Consider the solution of the equation (2) satisfying the initial condition

$$u(x,0) = e^{-x^2} \qquad x \in R \qquad (3)$$

We take up here a simple geometrical construction of the successive shapes of the initially single humped pulse given by (3). The graph of the solution at any time t (i.e., the pulse at time t) is obtained by translating a point P on the pulse (3) by a distance in positive x-direction, the magnitude of the translation being equal to t times the amplitude of the pulse at the point P.

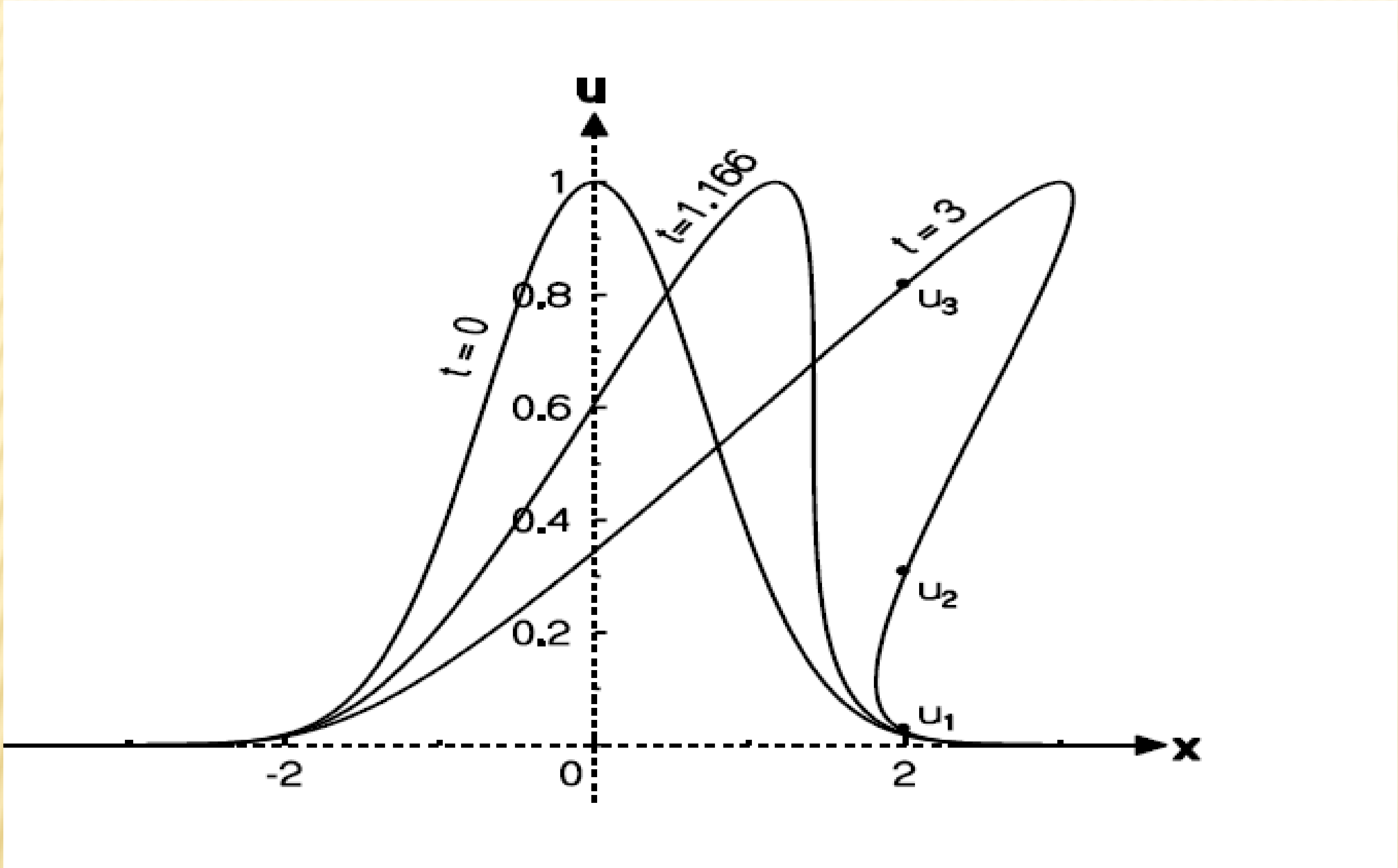

Fig. 1 As t increases, the pulse of the nonlinear wave deforms.

It has been observed in nature that a moving discontinuity appears in the quantity u immediately after the time t_c. This discontinuity at a point x = X(t) is called a shock.

When a shock appears in the solution, it fits into the multi-valued part of the solution in such a way that it cuts off lobes of areas on two sides of it in a certain ratio from the graph of the solution at any time t > t_c and makes the solution single valued. The ratio in which the lobes on the two sides are cut off depends on a more primitive property (conservation of an appropriate density) of the physical phenomena represented by the equation (2) . When the primitive property is a conservation of the density ρ(u) = u, the shock cuts off lobes of an equal area on the two sides of it.

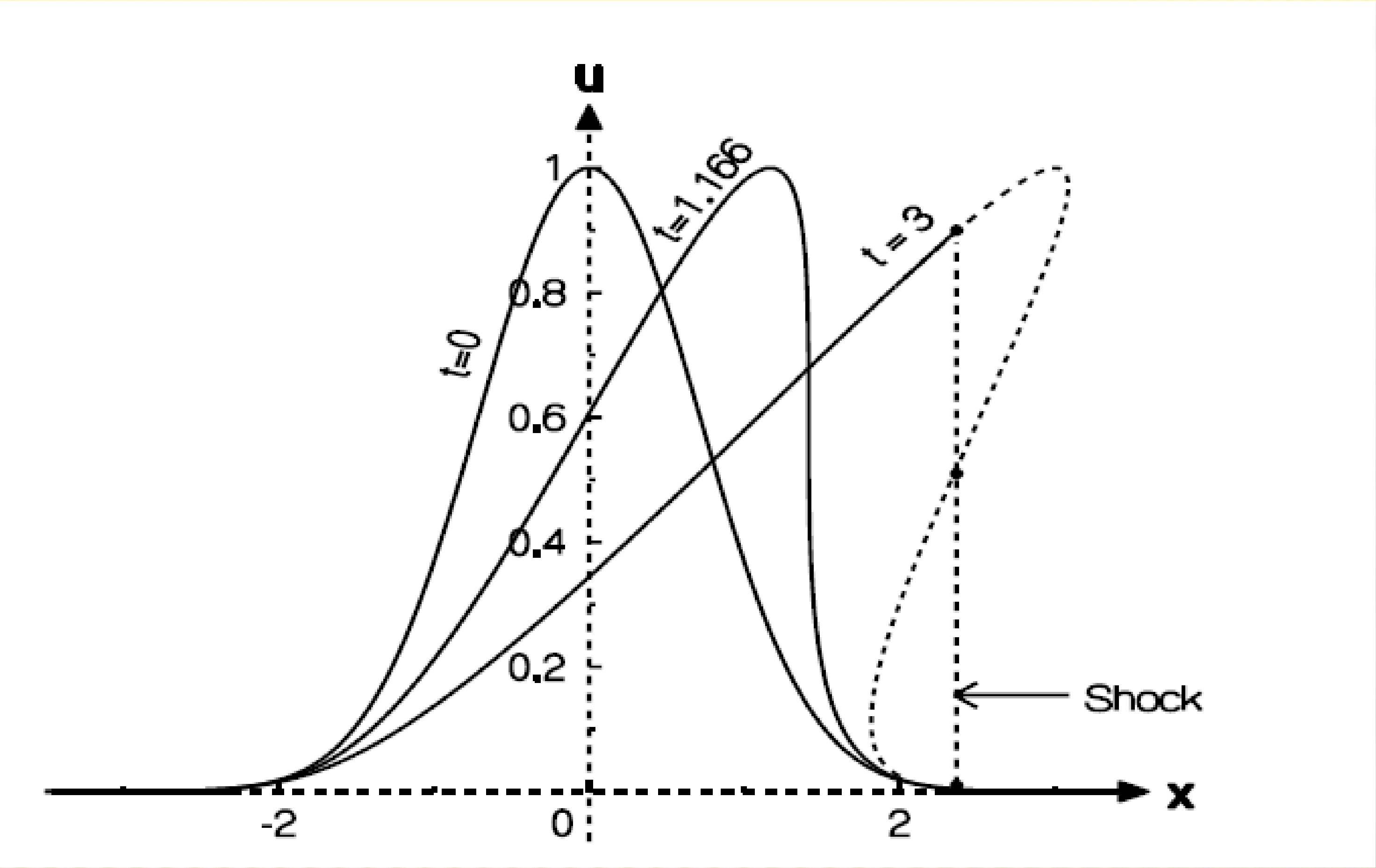

Fig.2 The shock (shown by broken vertical line) fits into the multi-valued part of the curve at t = 2 assuming that the shock cuts off lobes of equal areas on two sides of it.

References

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Thank You