Schmidt's Solution of non-homogeneous Fredholm Integral Equation of the second Kind

There are three important cases

Case I → Unique Solution

If $\lambda \neq \lambda_m \quad \forall\ m \in N$

then $a_m = \dfrac{\lambda}{\lambda_m - \lambda} f_m \quad (\lambda_m \neq \lambda)$

we can find the well defined value of a_m

Put this value of a_m in $y(x) - f(x) = \sum\limits_{m=1} a_m \varphi_m(x)$

Hence Solution Exists finitely

given by $y(x) = f(x) + \lambda \sum\limits_m \dfrac{f_m}{\lambda_m - \lambda} \varphi_m(x)$

if and only if λ does not take on an Eigen value.

Case II → No Solution

If λ_K be the K^{th} eigen value then,

$\lambda = \lambda_K$ and $f_K \neq 0$

i.e $\displaystyle\int_a^b f(x)\, \varphi_K(x)\, dx \neq 0$

i.e $\varphi_K(x)$ is not orthogonal to $f(x)$

Therefore We find that no Solution exists,

Since the term $\dfrac{f_K\, \varphi_K(x)}{\lambda_K - \lambda}$ is not defined.

Case III → Infinitely Many Solutions exists

let $\lambda = \lambda_K$ where λ_K is the K^{th} eigen Value and also let

$$f_K = 0 \quad i.e \int_a^b f(x)\, \varphi_K(x) = 0$$

i.e $\varphi_K(x)$ is orthogonal to $f(x)$

Then for $m = K$

$$C_m = f_m + \frac{\lambda C_m}{\lambda_m} \text{ given}$$

$$C_K = \underbrace{f_K}_{=0} + \frac{\cancel{\lambda}}{\cancel{\lambda_m}} C_K \qquad \because f_K = 0$$

$C_K = C_K$ which is a trivial Identity.

Therefore from $a_m = \frac{\lambda}{\lambda_m - \lambda} f_m \quad (\lambda_m \neq \lambda)$

the Coefficient a_K of $\varphi_K(x)$ in $y(x) = f(x) + \lambda \sum_m \frac{f_m}{\lambda_m - \lambda} \varphi_m(x)$

which formally assumes the form $\frac{0}{0}$

$$\boxed{\frac{\lambda f_K}{\lambda_K - \lambda} \quad \because \lambda_K = \lambda \quad \frac{0}{0} \quad \& \ f_K = 0}$$

is truely arbitrary Hence, in this Case solution

$$y(x) = f(x) + \lambda \sum_m \frac{f_m}{\lambda_m - \lambda} \varphi_m(x)$$ can be written as

$$y(x) = f(x) + A\varphi_K(x) + \lambda \sum_m \frac{f_m}{\lambda_m - \lambda} \varphi_m(x) \quad (m \neq K)$$

where A is any Constant.