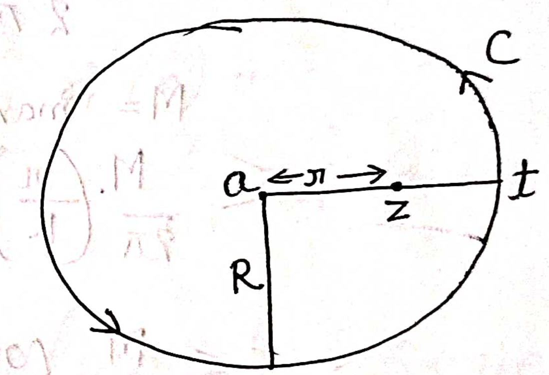
TAYLOR'S THEOREM → If a function $f(z)$ is analytic within a Circle C with its centre $z=a$ and radius R, then at every point z inside C

$$f(z) = \sum_{n=0}^{\infty} f^{(n)}(a) \frac{(z-a)^n}{n!} \quad \text{i.e} \quad f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

$$\text{where } a_n = \frac{f^{(n)}(a)}{n!}$$

[The series on the right hand side is known as Taylor's series of $f(x)$]

Proof ⇒ let $f(z)$ be analytic within a Circle C whose Equation is $|t-a| = R$




let z be any point within C

s.t. $|z-a| = r < R$

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(t)\, dt}{t-z}$$

$$t-z-a+a$$
$$(t-a)-(z-a)$$

$$= \frac{1}{2\pi i} \int_C \frac{f(t)\, dt}{(t-a)-(z-a)}$$

$$= \frac{1}{2\pi i} \int_C \frac{f(t)\, dt}{t-a} \left[1 - \left(\frac{z-a}{t-a}\right)\right]^{-1}$$

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(t)}{t-a} \left[1 + \frac{z-a}{t-a} + \left(\frac{z-a}{t-a}\right)^2 + \cdots + \left(\frac{z-a}{t-a}\right)^n + \left(\frac{z-a}{t-a}\right)^{n+1} \left(1 - \frac{z-a}{t-a}\right)^{-1}\right] dt$$

For

$$= (1-b)^{-1} = 1 + b + b^2 + \cdots + b^n + \frac{b^{n+1}}{1-b}$$

Using the formula

$$\frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i}\int_C \frac{f(t)\,dt}{(t-a)^{n+1}}$$

$$f(z) = f(a) + (z-a)\frac{f'(a)}{1!} + (z-a)^2\frac{f''(a)}{2!} + \cdots + (z-a)^n\frac{f^{(n)}(a)}{n!} + U_{n+1}$$

where $U_{n+1} = \frac{(z-a)^{n+1}}{2\pi i}\int_C \frac{f(t)\,dt}{(t-z)(t-a)^{n+1}}$

Now

$$|U_{n+1}| \leq \frac{|z-a|^{n+1}}{2\pi}\int_C \frac{|f(t)|\,|dt|}{(|t-a|-|z-a|)\,|t-a|^{n+1}}$$

$M = \max |f(t)|$ on $C$

$$\leq \frac{M}{2\pi}\left(\frac{r}{R}\right)^{n+1}\cdot\frac{1}{R-r}\cdot\int_C |dt|$$

$$\leq \frac{M}{\cancel{2\pi}}\left(\frac{r}{R}\right)^{n+1}\frac{1}{R-r}\cdot\cancel{2\pi} R$$

$|t-z|$

$(t-a+a-z)$

$= (t-a)-(z-a)$

$|t-a| = R$

$|z-a| = r$

$$|U_{n+1}| \leq M\cdot\left(\frac{r}{R}\right)^{n+1}\frac{1}{1-\left(\frac{r}{R}\right)} \to 0 \text{ as } n\to\infty$$

as since $\lim_{n\to\infty}\left(\frac{r}{R}\right)^{n+1} = 0$ as $\frac{r}{R}<1$

$\boxed{\text{or } \to 0 \text{ as } r \to 0}$

so $\lim_{n\to\infty} U_{n+1} = 0$

Hence

$$f(z) = \lim_{n\to\infty}\left[f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \cdots + \frac{(z-a)^n}{n!}f^{(n)}(a) + \cdots\right]$$

or $f(z) = \sum_{n=0}^{\infty}\frac{(z-a)^n}{n!}f^{(n)}(a) = \sum_{n=0}^{\infty} a_n(z-a)^n$

where $a_n = \frac{f^{(n)}(a)}{n!}$