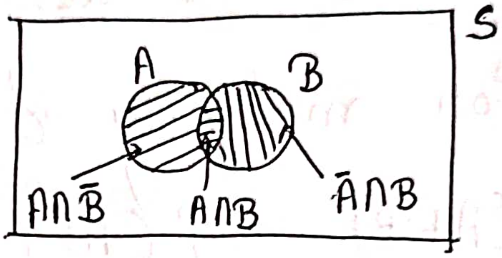
Addition Theorem of Probabilities (or Theorem of Total Probability)

Statement → If A and B are any two events then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Proof →



If A and $\bar{A} \cap B$ are disjoint sets and their union is $A \cup B$

$$\Rightarrow A \cup B = A \cup (\bar{A} \cap B)$$

$$\Rightarrow P(A \cup B) = P[A \cup (\bar{A} \cap B)]$$

$$= P(A) + P(\bar{A} \cap B)$$

$$= P(A) + [P(\bar{A} \cap B) + \underbrace{P(A \cap B) - P(A \cap B)}_{\text{add \& subtract}}]$$

$$= P(A) + P[(\bar{A} \cap B) \cup (A \cap B)] - P[(A \cap B)]$$

$\because$ $\bar{A} \cap B$ and $A \cap B$ are disjoint

$[\because (\bar{A} \cap B) \cup (A \cap B)] = B$

$$= P(A) + P(B) - P(A \cap B)$$

$$\Rightarrow P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Note 1 → If A and B are two mutually disjoint events then $A \cap B = \phi$

so that $P(A \cap B) = P(\phi) = 0$

so $P(A \cup B) = P(A) + P(B)$

Note-2 $P(A \cup B)$ is also written as $P(A+B)$

Thus for mutually disjoint events A & B

$$P(A+B) = P(A) + P(B)$$

$P(A \cap B)$ is also written as $P(AB)$

Example → In a given race, the odds in favour of four horses A, B, C, D are 1:3, 1:4, 1:5, 1:6 respectively. Assuming that a dead heat is impossible. find the chance that one of them wins the race

Solution → let p_1, p_2, p_3, p_4 be the probabilities of winning of the horses A, B, C, D, respectively.

Since a dead heat (in which all the four horses cover same distance in same time) is not possible the events are mutually exclusive.

odds in favour of A are 1:3

$$\therefore p_1 = \frac{1}{1+3} = \frac{1}{4}$$

Similarly $p_2 = \frac{1}{5}$, $p_3 = \frac{1}{6}$ $p_4 = \frac{1}{7}$

If p is the chance that one of them wins then

$$p = p_1 + p_2 + p_3 + p_4$$

$$= \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$$

$$= \frac{319}{420}$$