CHY'S INEQUALITY →

Theorem → If $f(z)$ be analytic in a domain D and C be the circle defined by $|z-a| = R$ which lies entirely in D and if $|f(z)| \leq M$ on C then

$$|f^n(a)| \leq \frac{M\, n!}{R^n}$$

Proof →

$$f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^{n+1}}$$

$$|f^n(a)| = \left|\frac{n!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^{n+1}}\right|$$

$$\leq \frac{n!}{|2\pi i|}\int_C \frac{|f(z)|\,|dz|}{|(z-a)^{n+1}|}$$

$$\leq \frac{n!}{2\pi}\frac{M}{R^{n+1}}\int_0^{2\pi} R\,d\theta$$

$\because z-a = Re^{i\theta}$

$\therefore dz = iRe^{i\theta}d\theta$

$\therefore |dz| = |iRe^{i\theta}d\theta|$

$= R\,d\theta$

$$\leq \frac{n!}{\cancel{2\pi}}\frac{M}{R^{n+1}}\, R\cdot\cancel{2\pi}$$

$$\leq \frac{n!\,M}{R^n}$$

blem → If C be a closed contour containing the origin inside it, prove that

$$\frac{a^n}{n!} = \frac{1}{2\pi i}\int_C \frac{e^{az}}{z^{n+1}}\,dz$$

Solution → We have

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^{n+1}}$$

Put $a=0$ $\quad f^{(n)}(0) = \frac{n!}{2\pi i}\int_C \frac{f(z)dz}{z^{n+1}}$

Taking $f(z) = e^{az}$

$f'(z) = a e^{az}$ $\quad f''(z) = a^2 e^{az}$

$f^{(n)}(z) = a^n e^{az}$ $\quad$ so $f^n(0) = a^n e^0 = a^n$

then $f^{(n)}(0) = a^n \cancel{a} = \frac{n!}{2\pi i}\int_C \frac{e^{az}dz}{z^{n+1}}$

$$f^{(n)}(0) = a^n = \frac{n!}{2\pi i}\int_C \frac{e^{az}\,dz}{z^{n+1}}$$

or $$\frac{a^n}{n!} = \frac{1}{2\pi i}\int_C \frac{e^{az}dz}{z^{n+1}}$$