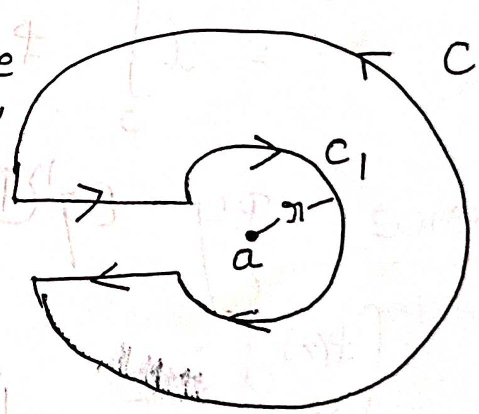
Cauchy's Integral Formula → If a function $f(z)$ is analytic within and on a closed curve C and a is any point within C, then

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{z-a}$$

Proof → Consider the function $\frac{f(z)}{z-a}$, which is analytic all all points within C except at $z=a$. with the point a as a centre and radius r, draw a circle c_1 lies entirely inside C.

Thus $\frac{f(z)}{z-a}$ is analytic in the region between C and C_1



so By Cauchy's Integral theorem for multiple connected region, We hav

$$\int_C \frac{f(z)\,dz}{z-a} = \int_{C_1} \frac{f(z)\,dz}{z-a} \longrightarrow (1)$$

After Eq^n ① or

$$\int_C \frac{f(z)\,dz}{z-a} = \int_{C_1} \frac{f(z)\,dz}{z-a} \rightarrow ①$$

Now, the equation of circle C_1 is $|z-a| = r$

or $z-a = re^{i\theta}$

$dz = ire^{i\theta}\,d\theta$

$$\int_{C_1} \frac{f(z)dz}{z-a} = \int_0^{2\pi} \frac{f(a+re^{i\theta})}{\cancel{re^{i\theta}}} i\cancel{re^{i\theta}}\,d\theta$$

$$= i\int_0^{2\pi} f(a+re^{i\theta})\,d\theta$$

Hence By Eq^n ① we have

$$\int_C \frac{f(z)d}{z-a} = i\int_0^{2\pi} f(a+re^{i\theta})\,d\theta \rightarrow ②$$

In the limiting form, as the circle C_1 shrinks to the point a ie $r \to 0$

then from ②

$$\int_C \frac{f(z)dz}{z-a} = i\int_0^{2\pi} f(a)\,d\theta = if(a)\int_0^{2\pi} d\theta$$

$$= if(a)\,[\theta]_0^{2\pi}$$

$$= 2\pi i f(a)$$

Hence $f(a) = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)dz}{z-a}$ Proved.