... the characteristic equation of the matrix, where

$$A = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$ Hence find A^{-1}

Solution → The characteristic Equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} 4-\lambda & 3 & 1 \\ 2 & 1-\lambda & -2 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)\left[(1-\lambda)^2 + 4\right] - 3\left[2(1-\lambda) + 2\right] + 1\left[4 - (1-\lambda)\right] = 0$$

$$(4-\lambda)\left[\lambda^2 + 1 - 2\lambda + 4\right] - 3(2 - 2\lambda + 2) + 1(4 - 1 + \lambda) = 0$$

$$(4-\lambda)(\lambda^2 - 2\lambda + 5) - 3(4 - 2\lambda) + 1(3 + \lambda) = 0$$

$$4\lambda^2 - 8\lambda + 20 - \lambda^3 + 2\lambda^2 - 5\lambda - 12 + 6\lambda + 3 + \lambda = 0$$

$$-\lambda^3 + 6\lambda^2 - 6\lambda + 11 = 0$$

$$\lambda^3 - 6\lambda^2 + 6\lambda - 11 = 0 \rightarrow (1)$$

By Cayley-Hamilton theorem

A must satisfy (1) i.e $A^3 - 6A^2 + 6A - 11I = 0$

Pre-multiplying both sides by A^{-1} we get

$$11A^{-1} = A^2 - 6A + 6I$$

$$11A^{-1} = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix} - 6\begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$11A^{-1} = \begin{bmatrix} 5 & -1 & -7 \\ -4 & 3 & 10 \\ 3 & -5 & -2 \end{bmatrix}$$

or $$A^{-1} = \frac{1}{11}\begin{bmatrix} 5 & -1 & -7 \\ -4 & 3 & 10 \\ 3 & -5 & -2 \end{bmatrix}$$

Here
$$A^2 = \begin{bmatrix} 23 & 17 & -1 \\ 8 & 3 & -2 \\ 9 & 7 & -2 \end{bmatrix}$$

so $A^3 - 11A^2 - \dots$

If $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ use Cayley Hamilton theorem to express $A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2$ as a linear Polynomial in A

Solution → The characteric Equation is

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ -1 & 3-\lambda \end{vmatrix}$$

$$\Rightarrow (1-\lambda)(3-\lambda) + 2 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda + 5 = 0 \rightarrow (1)$$

By Cayley- Hamilton theorem, the matrix A satisfy its characteristic Equation, we have

$$A^2 - 4A + 5I = 0$$

$$\text{or } A^2 = 4A - 5I \rightarrow (2)$$

Multiply both sides of (2) by A, A^2, A^3 & A^4 respectively we have

$$A^3 = 4A^2 - 5A$$
$$A^4 = 4A^3 - 5A^2$$
$$A^5 = 4A^4 - 5A^3$$
$$A^6 = 4A^5 - 5A^4$$

Now By given Expression $A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2$

Substitute for A^6 $\quad (\cancel{4A^5} - 5A^4) - \cancel{4A^5} + 8A^4 - 12A^3 + 14A^2$

$$= 3A^4 - 12A^3 + 14A^2$$

Substitute for A^4) $= 3(\cancel{4A^3} - 5A^2) - \cancel{12A^3} + 14A^2$

$$= -A^2$$

Substitute for A^2) $= -(4A - 5I)$

$= -4A + 5I$, which is the required linear polynomial in A.