Change of Interval and Functions having arbitrary period → OR Fourier Series of arbitrary Interval $(0, 2L)$ or $(-L, L)$

Fourier series of Periodic function with period 2π has already discussed as

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

The period of function is not always 2π but T or $2L$ This period must be converted to the length 2π ∴ The Independent variable x is also to be changed proportionally

Now we consider fourier series expansion of a function having period $2L$.

let $f(x)$ be a periodic function with period $2L$ defined in the Interval $-L < x < L$ or $f(x)$ be defined in the interval $(-L, L)$

Now we want to change the func
to the period 2π.

$\because$ $2L$ is the interval for variable x

$\therefore$ 1 " " " " " $= \frac{x}{2L}$

$\therefore$ 2π " " " " " $= \frac{x}{2L} \times 2\pi$

$= \frac{\pi x}{L}$

so put $t = \frac{\pi x}{L}$

or $x = \frac{Lt}{\pi}$

Thus the function $f(x)$ of period $2L$ is transformed to the function $f\left(\frac{Lt}{\pi}\right)$ or $F(t)$ of period 2π.

~~F(t) can be expanded in the Fourier Series~~

$\cancel{F(t) = f\left(\frac{Lt}{\pi}\right) = \frac{a_0}{2} +}$

so $f(x) = f\left(\frac{Lt}{\pi}\right) = F(t) \rightarrow (*)$

Thus $F(t)$ is defined in $-\pi \le t \le \pi$ and hence the Fourier series for $F(t)$ can be written as

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nt + b_n \sin nt\right.$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} F(t)\,dt$ $\quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(t)\cos nt\,dt$, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(t)\sin n$

$) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \Rightarrow ①$

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} F(t)\, dt \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(t) \cos nt\, dt \Rightarrow ②$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(t) \sin nt\, dt$

To obtain the Fourier series for $f(x)$ in $(-L, L)$ we change the variable by replacing t by $\frac{\pi x}{L}$ in ① & ②

$\therefore F(t) = f\left(\frac{Lt}{\pi}\right) = f\left(\frac{L}{\pi}\frac{\pi x}{L}\right)$ so using ⊛ Eq^n $F(t) = f\left(\frac{Lt}{\pi}\right) = f(x)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}\right) \Rightarrow ③$$

where
$$\left.\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)\, dx \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx \end{aligned}\right\} \Rightarrow ④$$

Also the function $f(x)$ might have been defined in the interval $(0, 2L)$ & Eq^n ④ would change

$$a_0 = \frac{1}{L}\int_0^{2L} f(x)\, dx, \quad a_n = \frac{1}{L}\int_0^{2L} f(x) \cos \frac{n\pi x}{L}\, dx$$

$$b_n = \frac{1}{L}\int_0^{2L} f(x) \sin \frac{n\pi x}{L}\, dx$$