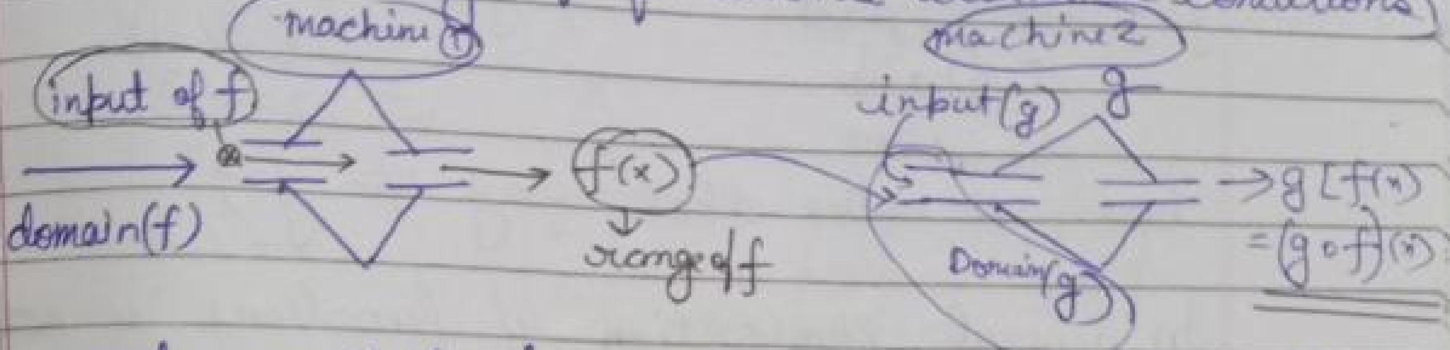
Compositions of Functions

(Combining of functions with some conditions)



~~out~~ output of machine 1 $\subseteq$ input of machine 2

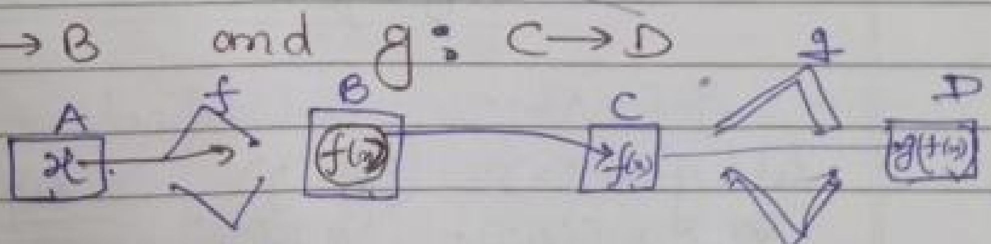
$$\boxed{\text{Range}(f) \subseteq \text{Domain}(g)}$$

\# $g \circ f\,(x)$ is defined if $\text{Range}(f) \subseteq \text{Domain}(g)$

$f \circ g\,(x)$ is defined if $\text{Range}(g) \subseteq \text{Domain}(f)$.

\# Let $f : A \rightarrow B$ and $g : C \rightarrow D$

$g \circ f : A \rightarrow D$



$g \circ f : A \rightarrow D$ is defined if $B \subseteq C$ or

range(f) $\subseteq$ Domain g

$f \circ g : C \rightarrow B$ is defined if range(g) $\subseteq$ Domain(f)

\# Let $f : A \rightarrow B$, $g : B \rightarrow A$

is $f \circ g$ defined? yes because Range(g) $\subseteq A =$ Domain(f)

is $g \circ f$ defined? yes because Range(f) $\subseteq B =$ Domain g

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Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$

$f(x) = x^2$ $\quad$ $g(x) = 2x+1$

find $g \circ f$? $\quad$ find $f \circ g$?

$$g \circ f(x) = g[f(x)] = g[x^2] = 2x^2+1$$

$$f \circ g(x) = f[g(x)] = f[2x+1] = (2x+1)^2$$

here we notice that $f \circ g \neq g \circ f$

In general composition of functions is not comutative.

Statement: Let $f: A \to B$ ang $g: B \to C$ be two functions.

Is $g \circ f$ defined.

~~(i) If $g \circ f : A \to C$ are one one,~~

★ Theorem: $\boxed{\text{If } f \text{ and } g \text{ are one one then } g \circ f \text{ is also one one.}}$

$g \circ f : A \to C$ is defined

Let $(g \circ f)(x) = (g \circ f)(y)$ $\quad \forall x, y \in A$

$\Rightarrow g[f(x)] = g[f(y)]$

$\Rightarrow f(x) = f(y)$ $\quad$ ($\because$ g is one one)

$\Rightarrow x = y$ $\quad$ ($\because$ f is one one)

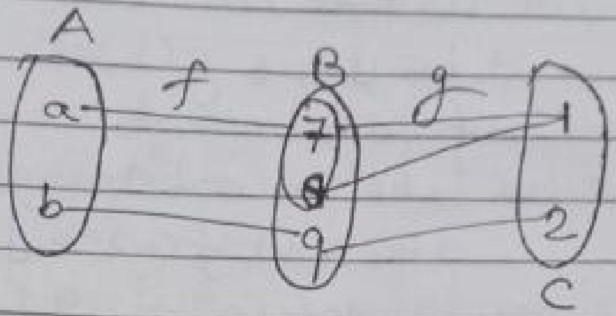
$\therefore$ $g \circ f$ is one one.

Statement: If f and g are 1-1, then $g \circ f$ is also 1-1.
(True)

Converse: If $g \circ f$ is 1-1 then f and g are 1-1.
(need not be true)

Counter:

$g \circ f : A \to C$



$g \circ f(a) = 1$

$g \circ f(b) = 2$ $\quad g \circ f$ is one one.

but f is one one and g is not one one.

Contrapositive: $\boxed{\text{If } g \circ f \text{ is not one one then either } f \text{ is not 1-1 or } g \text{ is not 1-1.}}$

Trick to write counter for Converse:

Step ① write $g \circ f \to$ (one one)

Step ② write f one one.

now show g is not one one.

Counter: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$ $\qquad g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2$

$g \circ f : \mathbb{R} \to \mathbb{R}$

$$g[f(x)] = g[e^x] = (e^x)^2 = e^{2x}$$

$g \circ f$ is one one, but f is one one and g is not one-one.

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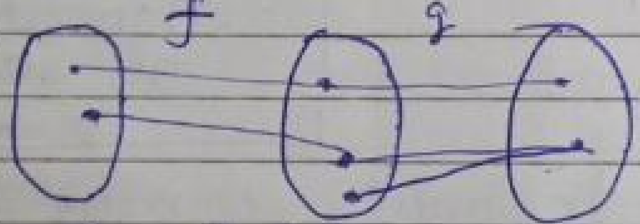
Thm: If f and g are onto then $g \circ f$ is also onto

Pf
Let $f: A \to B$ and $g: B \to C$
g is onto
Let us take an arbitrary $c \in C$,
$\exists\, b \in B$ s.t $g(b) = c$
$\because f$ is also onto
Let us take an arbitrary element $b \in B$,
$\exists\, a \in A$ s.t $f(a) = b$
now $g \circ f(a) = g[f(a)]$
$= g[b]$
$= c$
$\therefore g \circ f$ is onto

Converse need not be true.
Counter:



$g \circ f$ onto, but f is not onto

Contrapositive: If $g \circ f$ is not onto then either g is not onto or f is not onto.

Thm: If $g \circ f$ is onto then g is onto.
$f: A \to B$, $g: B \to C$
given; $g \circ f: A \to C$ is onto
$\forall\, c \in C$, $\exists$ atleast one $a \in A$ s.t
$g \circ f(a) = c$
$g[f(a)] = c$ ⊛
$\therefore g$ is onto as $\forall\, c \in C$, $\exists\, f(a) \in B$ s.t ⊛

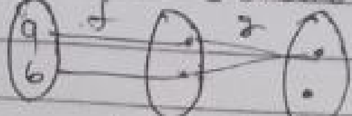
Prefactor

If $\boxed{g \circ f}$ is 1-1 then f is 1-1

post factor

$\boxed{\text{If Composition is one-one then postfactor is always one-one.}}$

Converse If f is one-one then $g \circ f$ is one-one. (need not be true)

Counter:- 

If $g \circ f$ is bijective, what can you say about f?

f is one-one.

Statement:- If Composition is onto then prefactor is always onto.

i.e if $g \circ f$ is onto then g is onto

Pf:- $f: A \to B$ and $g: B \to C$

$g \circ f$ is onto

$\Rightarrow \forall\, y \in B,\ \exists\, x \in A$ s.t $(g \circ f)(x) = y$

$\Rightarrow g[f(x)] = y \quad \forall\, y$

$\therefore \forall y \in B, \exists f(x) \in B$ s.t $g[f(x)] = y$ $\therefore$ g is onto.

Converse need not be true?

If $g \circ f$ is bijective then g is onto

If Composition is bijective then post factor is one-one and prefactor is onto.

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Result: If $g \circ f$ is one-one and f is onto, then g is one-one.

$f : A \to B$, $g : B \to C$

$g \circ f : A \to C$

$\forall y \in B, \exists x \in A$

$f(x) = y$

$g \circ f(x) = g[y] \in C$

$g \circ f(x_1) = g\,y(x_2)$

$g(y_1) = g(y_2)$

g

$g($

Result: If $g \circ f$ is onto and g is 1-1, then f is onto

\#

Results:

1. If $g \circ f$ is bijective, then f is one-one

2. If Composition is onto then prefactor is always onto

3. If $g \circ f$ is bijective then g is onto

4. If $g \circ f$ is bijective then f is one-one and g is onto.

5. If $g \circ f$ is one-one and f is onto then g one-one

6. If $g \circ f$ is onto and g is one-one then f is ~~one-one~~ onto.