sing Contour integration, Evaluate $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$ where $a > |b|$

Hence or otherwise evaluate

(i) $\int_0^{2\pi} \frac{d\theta}{\sqrt{2}-\cos\theta}$ (ii) $\int_0^{\pi} \frac{d\theta}{a+b\cos\theta}$; $a > |b|$

<u>Solution</u> Consider the integration round a unit circle $C \equiv |z| = 1$ so that $z = e^{i\theta}$

$$dz = ie^{i\theta}d\theta = i.z d\theta$$

$$d\theta = \frac{dz}{iz}$$

Also $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}\left(z+\frac{1}{z}\right)$

Then the given integral reduces to

$$I = \oint_C \frac{1}{\left[a+\frac{b}{2}\left(z+\frac{1}{z}\right)\right]}\left(\frac{dz}{iz}\right)$$

$$= \oint_C \frac{1}{a+\frac{b}{2z}(z^2+1)}\left(\frac{dz}{iz}\right) = \oint_C \frac{2z}{bz^2+2az+b}\left(\frac{dz}{iz}\right)$$

$$= \frac{2}{ib}\oint_C \frac{dz}{z^2+\frac{2a}{b}z+1}$$

$$= \frac{2}{ib}\oint \frac{dz}{(z-\alpha)(z-\beta)}$$

Poles are given by

$(z-\alpha)(z-\beta) = 0 \Rightarrow z = \alpha, \beta$

Both are simple poles

so $\alpha = -\frac{a}{b} + \frac{\sqrt{a^2-b^2}}{b}$

$\beta = -\frac{a}{b} - \frac{\sqrt{a^2-b^2}}{b}$

where α & β $= \frac{-\frac{2a}{b} \pm \sqrt{\frac{4a^2}{b^2}-4.1.1}}{2.1}$

$= \frac{-\frac{2a}{b} \pm \sqrt{4a^2-4b^2/b^2}}{2}$

$= \frac{\cancel{2}}{\cancel{2}}\left[\frac{-a}{b} \pm \frac{\sqrt{a^2-b^2}}{b}\right]$

Since $a > |b|$ $\quad \therefore |\beta| > 1$

Since $\alpha\beta = 1$

$|\alpha\beta| = 1$

$|\alpha||\beta| = 1$

$|\alpha| < 1 \quad \because |\beta| > 1$

Hence $z = \alpha$ is the only pole which lies inside the circle $C \equiv |z| = 1$

Residue of $f(z)$ at $(z = \alpha)$ is

$$R = \lim_{z \to \alpha} (z-\alpha) f(z)$$

$$= \lim_{z \to \alpha} \cancel{(z-\alpha)} \cdot \frac{2}{ib\cancel{(z-\alpha)}(z-\beta)}$$

$$= \frac{2}{ib(\alpha-\beta)} = \frac{\cancel{2}}{ib \cdot \frac{\cancel{2}\sqrt{a^2-b^2}}{\cancel{b}}}$$

$$= \frac{1}{i\sqrt{a^2-b^2}}$$

$$\alpha - \beta$$

$$-\frac{a}{b} + \frac{\sqrt{a^2-b^2}}{b} - \left[-\frac{a}{b} - \frac{\sqrt{a^2-b^2}}{b}\right]$$

$$= -\frac{a}{b} + \frac{\sqrt{a^2-b^2}}{b} + \frac{a}{b} + \frac{\sqrt{a^2-b^2}}{b}$$

$$= \frac{2\sqrt{a^2-b^2}}{b}$$

By Cauchy's Residue Theorem

$$I = 2\pi i(R) = 2\pi \cancel{i}\left(\frac{1}{\cancel{i}\sqrt{a^2-b^2}}\right)$$

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}} \quad \text{Ans (II)}$$

Using the property of Definite Integral

$$\cancel{2}\int_0^{\pi} \frac{d\theta}{a + b\cos\theta} = \frac{\cancel{2}\pi}{\sqrt{a^2-b^2}}$$

$$\int_0^{\pi} \frac{d\theta}{a + b\cos\theta} = \frac{\pi}{\sqrt{a^2-b^2}}$$

Put $a = \sqrt{2}$ and $b = -1$

$$\int_0^{2\pi} \frac{d\theta}{\sqrt{2} - \cos\theta} = \frac{2\pi}{\sqrt{2-1}} = 2\pi \quad \text{Ans (I)}$$