Reduction of Valterra Integral Equation of First Kind ~~into~~ to a Valterra Integral Equation of Second Kind

Proof → let $f(x) = \int_0^x K(x,t)\, u(t).dt \rightarrow (1)$

be the given Volterra Integral Equation of the first Kind, where $u(x)$ is an unknown function.

Let Kernel $K(x,t)$ and all its partial derivatives are Continuous, The necessary Condition for existence of Continuous solution is $f(0)=0$

Differentiate (1) w.r.t. x using Leibnitz rule

$$f'(x) = \int_0^x K'_x(x,t)\, u(t)\, dt + K(x,x) u(x) \frac{dx}{dx} - K(x,0)\, u(0) \frac{d0}{dx}$$

$$\text{or } f'(x) = \int_0^x K'_x(x,t)\, u(t)\, dt + K(x,x)\, u(x) \rightarrow (2)$$

Conversely, every solution of (2) satisfies (1) since the two member are zero at $x=0$ and their derivatives are Identical

If $K(x,x)$ does not Vanish at any point in the interval $[0,x]$ then Eq^n (2) can be written as

$$u(x) = \frac{f'(x)}{K(x,x)} - \int_0^x \frac{K'_x(x,t)}{K(x,x)}\, u(t)\, dt \rightarrow (3)$$

which reduces to a Volterra Integral Equation of Second Kind

If $K(x,x)$ vanishes then Eq ② again to an Equation of the first kind, which we can treat in a same fashion, provided that $K(x,t)$ possesses a continuous second derivative.

Differentiate ② w.r.t. x

$$f''(x) = K'_x(x,x)\,u(x) + \int_0^x K''_x(x,t)\,u(t)\,dt \rightarrow (9)$$

which is the Volterra Integral Equation of second kind provided $K'_x(x,x) \neq 0$

Again if $K'_x(x,x) = 0$ then we proceed as above.

Then we get a sequence of successive derivatives with regard to x of the Kernel $K(x,t)$ until we get a derivative $K^{n-1}(x,t) \neq 0$ for $x=t$

For continuous solution, it is necessary that $f'(x), f''(x) \text{ ------ } f^{(n-1)}(x)$ are all zero at $x=0$

Finally we get the Equation

$$K_x^{(n-1)}(x,a)\,u(x) + \int_0^x \frac{\partial^n}{\partial x^n} K(x,t)\,u(t)\,dt = f^{(n)}(x)$$

$$u(x) = \frac{f^{(n)}(x)}{K_x^{(n-1)}(x,x)} + \int_0^\infty \frac{K_x^n(x,t)}{K_x^{(n-1)}(x,x)}\,u(t)\,dt$$

which is an Equation of second kind provide $K^{n-1}(x,x) \neq 0$