... between a linear differential ...ation and a Fredholm Integral Equation

OR

Method of Converting a Boundary Value Problem to a Fredholm Integral Equation

Example → Deduce the following boundary value problem into an Integral Equation

$$\frac{d^2u}{dx^2} + \lambda u = 0 \qquad u(0) = 0$$
$$u(l) = 0$$

Solution → $\frac{d^2u}{dx^2} = -\lambda u$

Integrate Both Sides w.r.t x from 0 to x

$$\frac{du}{dx} - u'(0) = -\lambda \int_0^x u(x)\, dx$$

let $u'(0) = b$

$$\frac{du}{dx} = b - \lambda \int_0^x u(x)\, dx$$

Again Integrate w.r.t x from 0 to x

$$u(x) - u(0) - bx - \lambda \int_0^x u(x)(dx)^2$$

$u(0) = 0$ given

$$u(x) = bx - \lambda \int_0^x (x-t)\, u(t)\, dt \rightarrow (1)$$

Putting $x = l$ in above Equation

$$u(l) = bl - \lambda \int_0^l (l-t)\, u(t)\, dt$$ given $u(l)=0$

But $u(l) = 0$

so $bl = \lambda \int_0^l (l-t)\, u(t)\, dt$

$$b = \frac{\lambda}{l} \int_0^l (l-t)\, u(t)\, dt$$

Put the value of b in ①

$$u(x) = \frac{\lambda x}{l} \underbrace{\int_0^l}_{\int_0^x + \int_x^l} (l-t)\, u(t)\, dt - \lambda \int_0^x (x-t)\, u(t)\, dt$$

$$u(x) = \frac{\lambda}{l} \int_0^x x(l-t)\, u(t)\, dt + \frac{\lambda}{l} \int_x^l x(l-t)\, u t\, dt$$

$$- \lambda \int_0^x (x-t)\, u(t)\, dt$$

$$u(x) = \int_0^{\cancel{x}x} \underbrace{\left[\frac{\lambda x(l-t)}{l} - \lambda(x-t)\right]}_{\frac{\cancel{\lambda x l} - \lambda x t - \cancel{\lambda x l} + \lambda l t}{l}} u(t)\, dt + \int_x^l \frac{\lambda x}{l} (l-t)\, u(t)\, dt$$

$$u(x) = \int_0^x \frac{\lambda t(l-x)}{l}\, u(t)\, dt + \int_x^l \frac{\lambda x}{l} (l-t)\, u(t)\, dt$$

$$u(x) = \lambda \int_0^x K(x,t)\, u(t)\, dt$$

where
$$K(x,t) = \begin{cases} \frac{x}{l}(l-t) & 0 < x < t \\ \frac{t}{l}(l-x) & t < x < l \end{cases}$$