CONVOLUTION THEOREM → let $f_1(t)$ and $f_2(t)$ be two functions of t

and if $L[f_1(t)] = F_1(s)$ and $L[f_2(t) = F_2(s)]$

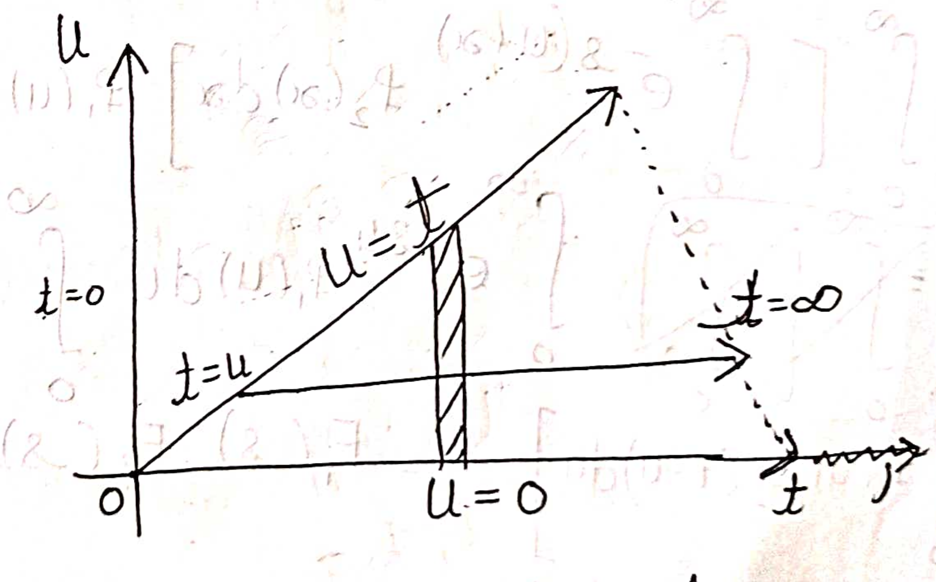
then $L\left[\int_0^t f_1(u) f_2(t-u)\,du = L\int_0^t f_2(u) f_1(t-u)\,du = F_1(s).F_2(s)\right.$

or $\boxed{L^{-1}[F_1(s).F_2(s)] = \int_0^t f_1(u) f_2(t-u)\,du = \int_0^t f_2(u) f_1(t-u)\,du}$

Proof → By Definition

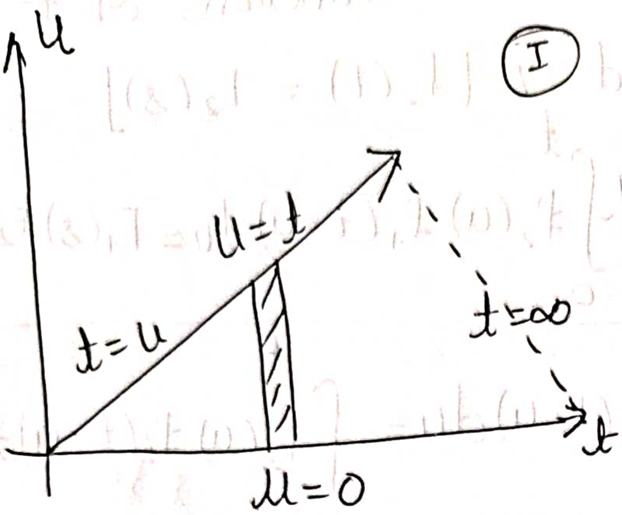
$$L\left[\int_0^t f_1(u) f_2(t-u)\,du\right] = \int_0^\infty e^{-st}\left[\int_0^t f_1(u) f_2(t-u)\,du\right] dt$$

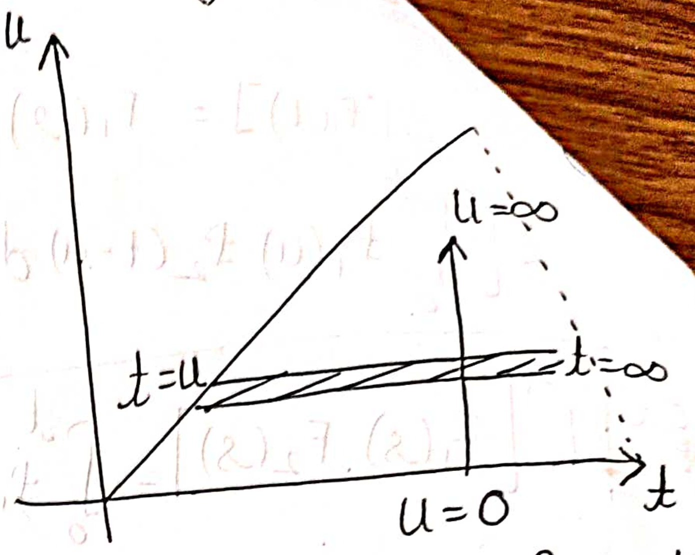
where the double integral is taken over the infinite region in the first quadrant lying between the lines ~~x~~ $u=0$ and $u=t$



Here first we integrate w.r.t 'u' within limits $u=0$ and $u=t$ and then we integrate w.r.t 't' with limits $t=0$ and $t=\infty$

on changing the order of integration





on changing the order of integration first we integrate w.r.t 't' with limits $t=u$ and $t=\infty$ and then w.r.t 'u' with limit $u=0$ and $u=\infty$

$$\int_{u=0}^{\infty} du \left[\int_{t=u}^{\infty} e^{-st} f_1(u)\, f_2(t-u)\, dt\right]$$

$$= \int_0^\infty \left[\int_u^\infty e^{-st} f_2(t-u)\,dt\right] f_1(u)\,du$$

Put $t-u=x$ in inner integral

change the limit from 0 to ∞

$$= \int_0^\infty \left[\int_0^\infty e^{-s(u+x)} f_2(x)\,dx\right] f_1(u)\,du$$

$$\cancel{= \int_0^\infty \left[\int_0^\infty\right.} \quad \int_0^\infty e^{-su} f_1(u)\,du \int_0^\infty e^{-sx} f_2(x)\,dx$$

$$L\left[\int_0^t f_1(u)\, f_2(t-u)\,du\right] = F_1(s)\, F_2(s)\,ds$$

Hence $L^{-1}\left[F_1(s)\, F_2(s)\right] = \int_0^t f_1(u)\, f_2(t-u)\,du$

The function $\int_0^t f_1(u)\, f_2(t-u)\, du$ is called the convalution of f_1 and f_2 and is denoted by $f_1 * f_2$

Thus $L[f_1 * f_2] = F_1(s)\, F_2(s)$. It is easy to verify that $f_1*f_2 = f_2*f_1$