Lecture of Module 2

Logic Gates

Overview

- **Introduction**
- **Logical Operators**
- **Basic Gates**
- **Universal Gates**
- **Realization of Basic Gates using Universal Gates**
- **Other Logic Gates**

Introduction

- Binary variables take on one of two values
- Logical operators operate on binary values and binary variables
- Basic logical operators are the logic functions AND, OR and NOT
- Logic gates implement logic functions
- Boolean Algebra: a useful mathematical system for specifying and transforming logic functions
- We study Boolean algebra as a foundation for designing and analyzing digital systems

Binary Variables

- **Recall that the two binary values have different names:**
  - **True/False**
  - **On/Off**
  - **Yes/No**
  - **1/0**
- **We use 1 and 0 to denote the two values.**
- **Variable identifier examples:**
  - **A, B, x, y, z, or X_1 , X_2 etc. for now**

Logical Operations

- The three basic logical operations are:
  - AND
  - OR
  - NOT
- AND is denoted by a dot (·)
- OR is denoted by a plus (+)
- NOT is denoted by an over bar (¯), a single quote mark (') after, or (~) before the variable

Operator

- Operators operate on binary values and binary variables
- Operations are defined on the values "0" and "1" for each operator:

AND	OR	NOT
$0 \cdot 0 = 0$	$0 + 0 = 0$	$\overline{0} = 1$
$0 \cdot 1 = 0$	$0 + 1 = 1$	$\overline{1} = 0$
$1 \cdot 0 = 0$	$1 + 0 = 1$	
$1 \cdot 1 = 1$	$1 + 1 = 1$	

Truth Tables

- ***Truth table*** - a tabular listing of the values of a function for all possible combinations of values on its arguments
- Example: Truth tables for the basic logic operations:

AND		
X	Y	$Z = X \cdot Y$
0	0	0
0	1	0
1	0	0
1	1	1

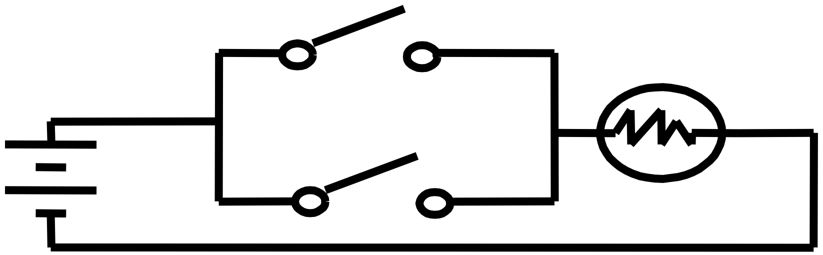
OR		
X	Y	$Z = X+Y$
0	0	0
0	1	1
1	0	1
1	1	1

NOT	
X	$Z = \overline{X}$
0	1
1	0

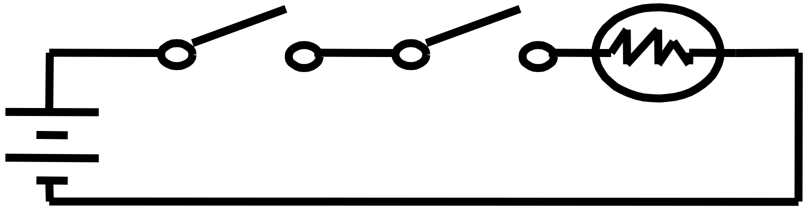
Logic Function Implementation

- Using Switches
  - For inputs:
    - logic 1 is switch closed
    - logic 0 is switch open
  - For outputs:
    - logic 1 is light on
    - logic 0 is light off.

Switches in parallel => OR



Switches in series => AND



Logic Gates

- In the earliest computers, switches were opened and closed by magnetic fields produced by energizing coils in *relays*. The switches in turn opened and closed the current paths.
- Later, *vacuum tubes* that open and close current paths electronically replaced relays.
- Today, *transistors* are used as electronic switches that open and close current paths.
- NOT, AND and OR Gates (Basic gates)
- NAND and NOR Gates (Universal logic gates)

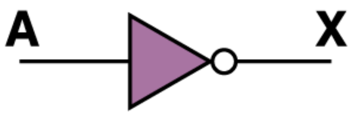
NOT Gate

A NOT gate accepts one input signal (0 or 1) and returns the opposite signal as output

Boolean Expression

$$X = A'$$

Logic Diagram Symbol



Truth Table

A	X
0	1
1	0

AND Gate

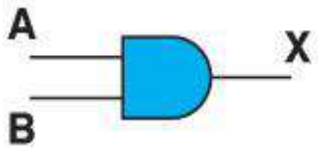
If all inputs are 1, the output is 1; otherwise, the output is 0

Or if any input is 0, output is 0

Boolean Expression

$$X = A \cdot B$$

Logic Diagram Symbol



Truth Table

A	B	X
0	0	0
0	1	0
1	0	0
1	1	1

OR Gate

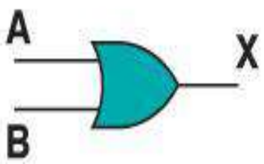
If all inputs are 0, the output is 0; otherwise, the output is 1

Or if any input is 1, output will be 1

Boolean Expression

$$X = A + B$$

Logic Diagram Symbol



Truth Table

A	B	X
0	0	0
0	1	1
1	0	1
1	1	1

Universal Gates

- Universal Logic Gate: Any basic gate or logic function can be realized using this gate
- Two universal logic gates
  - NAND
  - NOR

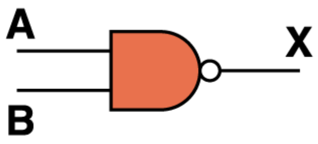
NAND Gate

If all inputs are 1, the output is 0; otherwise, the output is 1

Boolean Expression

$X = (A \cdot B)'$

Logic Diagram Symbol



Truth Table

A	B	X
0	0	1
0	1	1
1	0	1
1	1	0

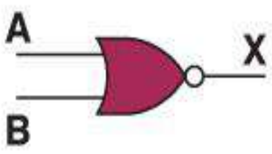
NOR Gate

If all inputs are 0, the output is 1; otherwise, the output is 0

Boolean Expression

$X = (A + B)'$

Logic Diagram Symbol

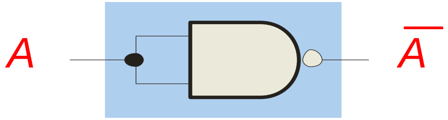


Truth Table

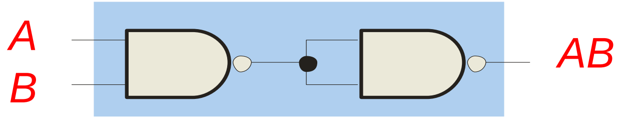
A	B	X
0	0	1
0	1	0
1	0	0
1	1	0

Realization

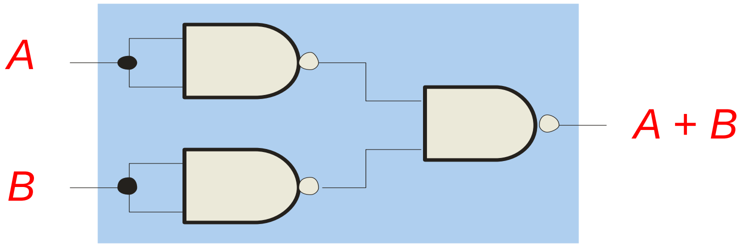
NAND gates are sometimes called **universal** gates because they can be used to produce the other basic Boolean functions.



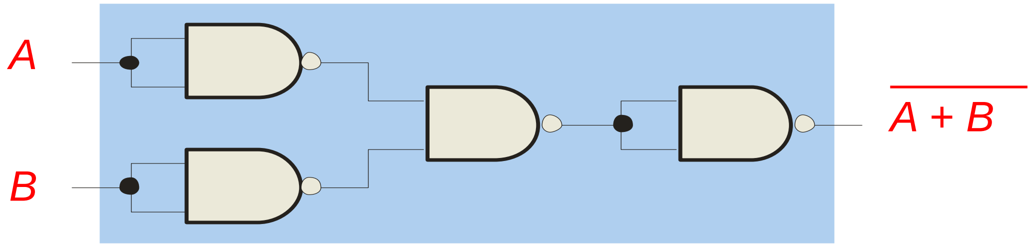
Inverter



AND gate



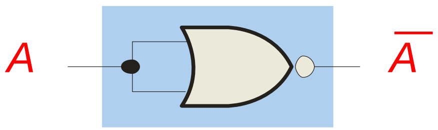
OR gate



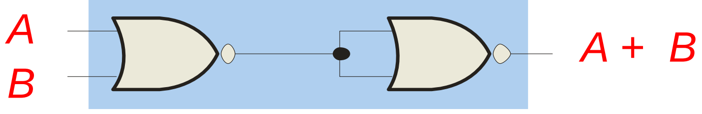
NOR gate

Realization

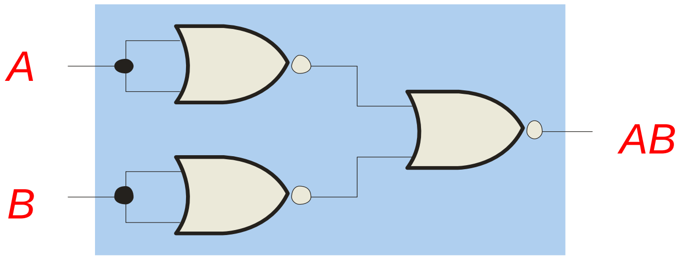
NOR gates are also **universal** gates and can form all of the basic gates.



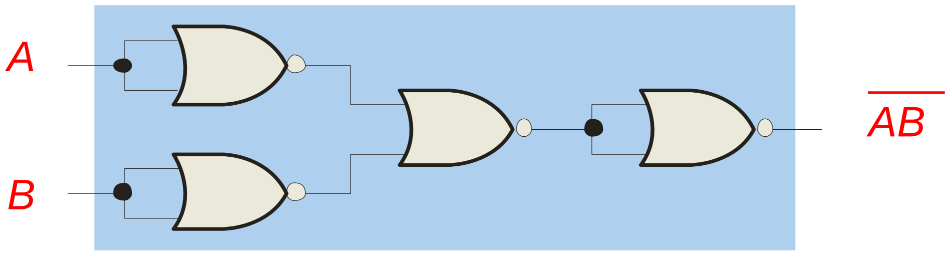
Inverter



OR gate



AND gate



NAND gate

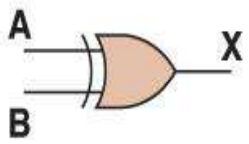
XOR Gate

If odd numbers of inputs are 1, the output is 1; otherwise, the output is 0

Boolean Expression

$$X = A \oplus B$$

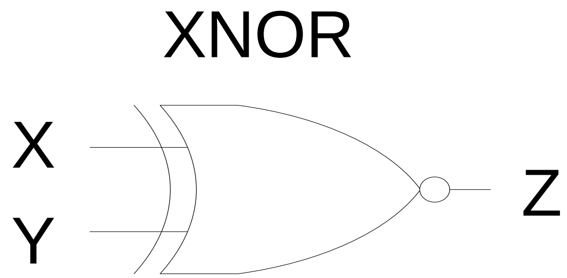
Logic Diagram Symbol



Truth Table

A	B	X
0	0	0
0	1	1
1	0	1
1	1	0

X-NOR Gate



X	Y	Z
0	0	1
0	1	0
1	0	0
1	1	1

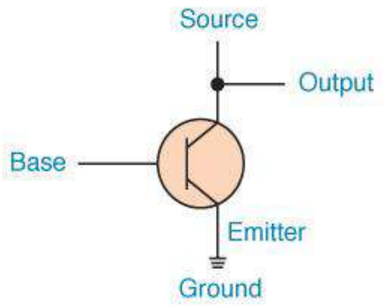
Constructing Gates

Transistor

A device that acts either as a wire that conducts electricity or as a resistor that blocks the flow of electricity, depending on the voltage level of an input signal

A transistor has no moving parts, yet acts like a switch

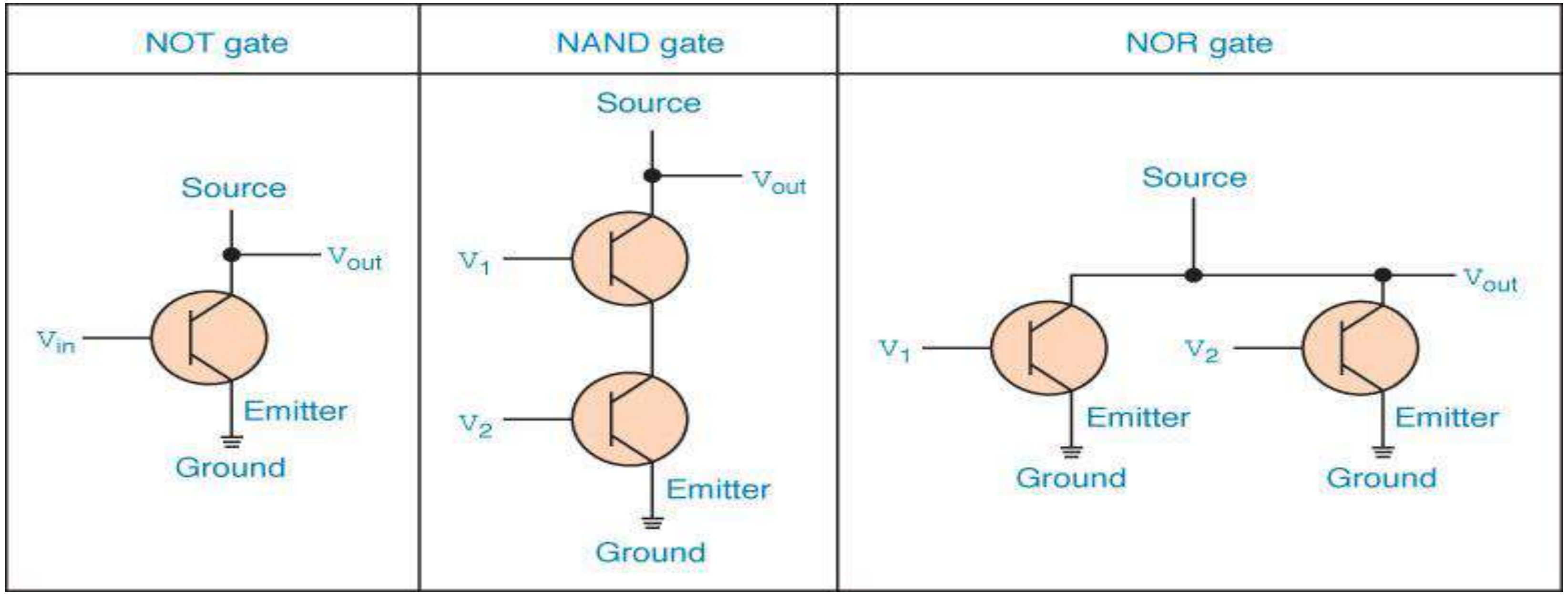
It is made of a semiconductor material, which is neither a particularly good conductor of electricity nor a particularly good insulator



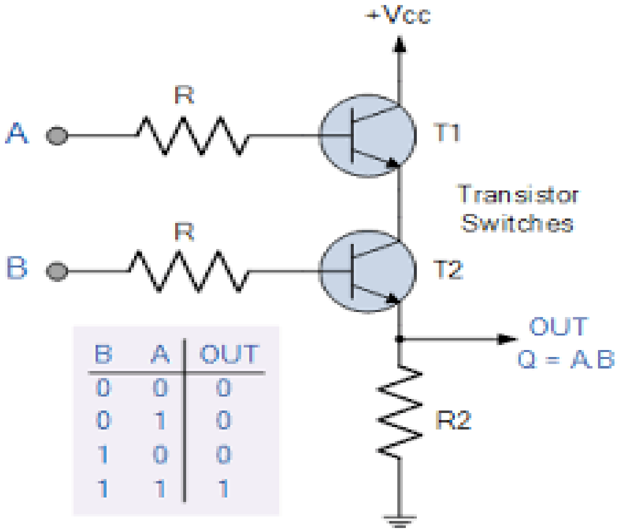
A transistor has three terminals

- A source
- A base
- An emitter, typically connected to a ground wire

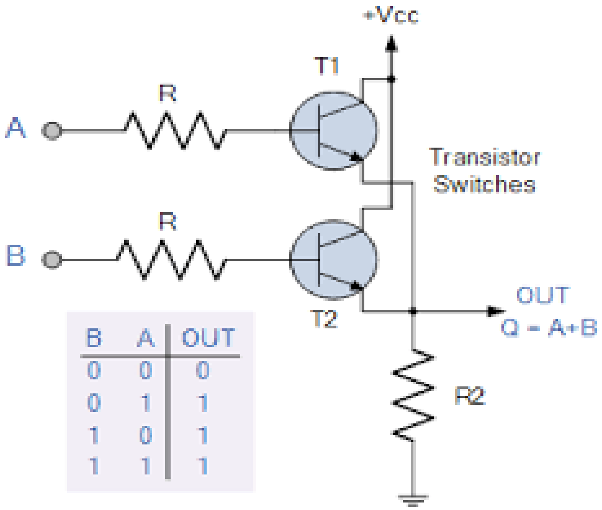
If the electrical signal is grounded, it is allowed to flow through an alternative route to the ground (literally) where it can do no harm


NOT gate
NAND gate
NOR gate
Source
Vout
Vin
Emitter
Ground
Source
Vout
V1
V2
Emitter
Ground
Source
Vout
V1
V2
Emitter
Emitter
Ground
Ground

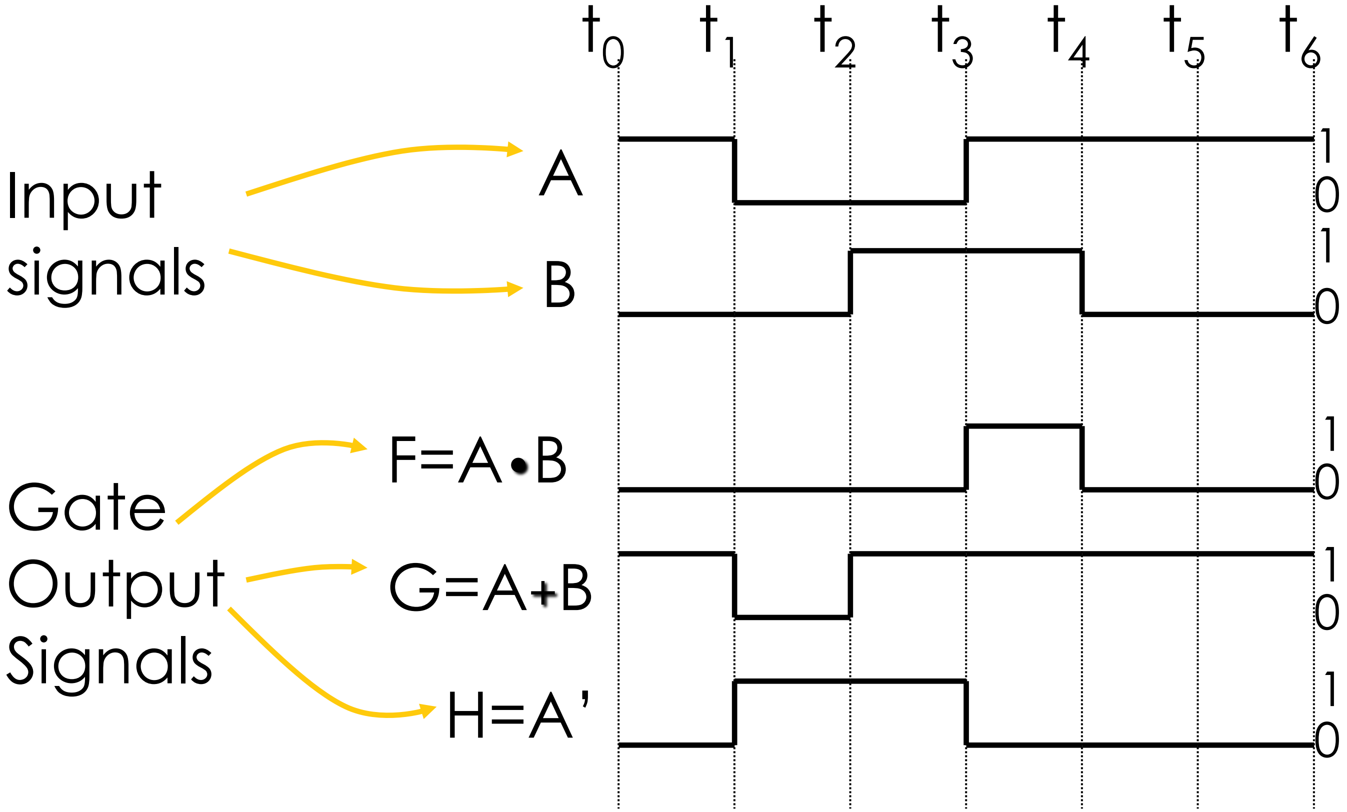
AND Gate



OR Gate



Timing Diagram

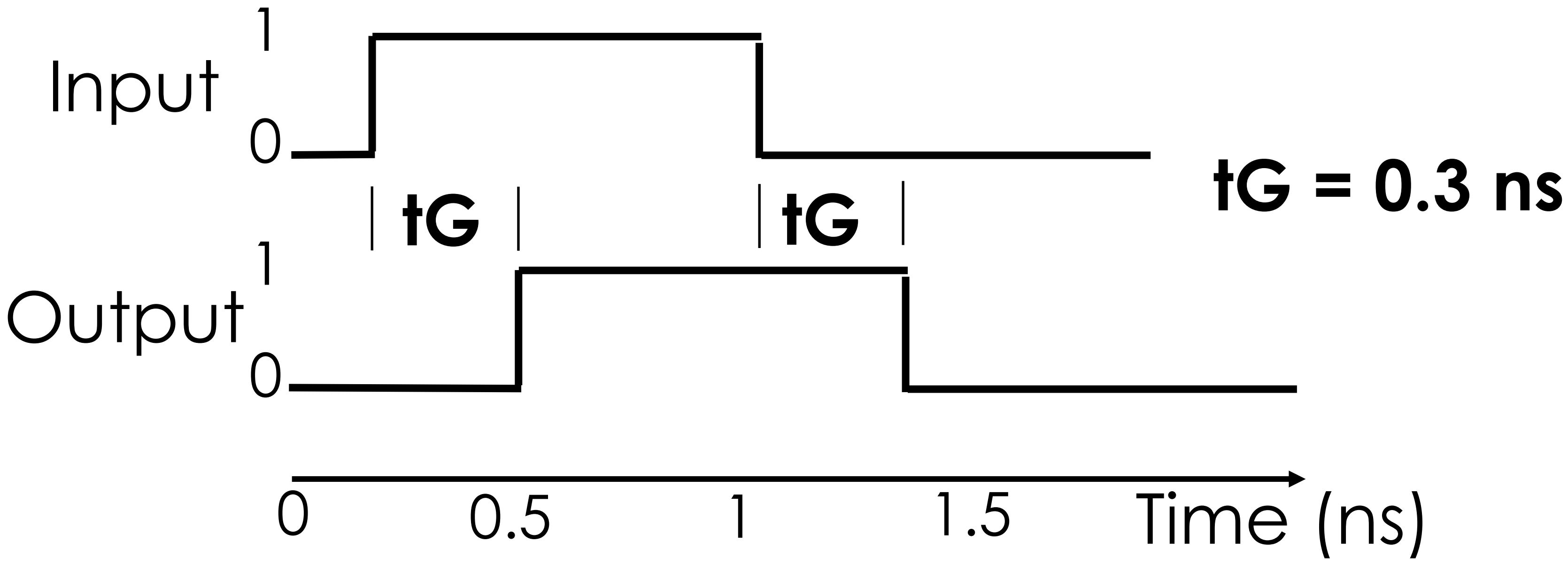


Transitions

<u>Basic Assumption:</u>
Zero time for signals to propagate Through gates

Gate Delay

- **In actual physical gates, if one or more input changes causes the output to change, the output change does not occur instantaneously.**
- **The delay between an input change(s) and the resulting output change is the *gate delay* denoted by t_G:**



Boolean Algebra

Overview

- Introduction
- Boolean Algebra
- Properties
- Algebraic Manipulation
- De-Morgan Theorem
- Complementation
- Truth Table

Introduction

- Understand the relationship between Boolean logic and digital computer circuits.
- Learn how to design simple logic circuits.
- Understand how digital circuits work together to form complex computer systems.
- In the latter part of the nineteenth century, **George Boole** suggested that logical thought could be represented through mathematical equations.
- Computers, as we know them today, are implementations of Boole's *Laws of Thought*.
- In this chapter, you will learn the simplicity that constitutes the essence of the machine (Boolean Algebra).

Boolean algebra

- Boolean algebra is a mathematical system for the manipulation of variables that can have one of two values.
  - In formal logic, these values are “true” and “false.”
  - In digital systems, these values are “on” and “off,” 1 and 0, or “high” and “low.”
- Boolean expressions are created by performing operations on Boolean variables.
  - Common Boolean operators include AND, OR, NOT, XOR, NAND and NOR

- A Boolean operator can be completely described using a truth table.
- The truth table for the Boolean operators AND, OR and NOT are shown at the right.
- The AND operator is also known as a Boolean product.
- The OR operator is the Boolean sum.
- The NOT operation is most often designated by an over-bar. It is sometimes indicated by a prime mark (‘) or an “elbow” (¬).

X AND Y

X	Y	XY
0	0	0
0	1	0
1	0	0
1	1	1

X OR Y

X	Y	X+Y
0	0	0
0	1	1
1	0	1
1	1	1

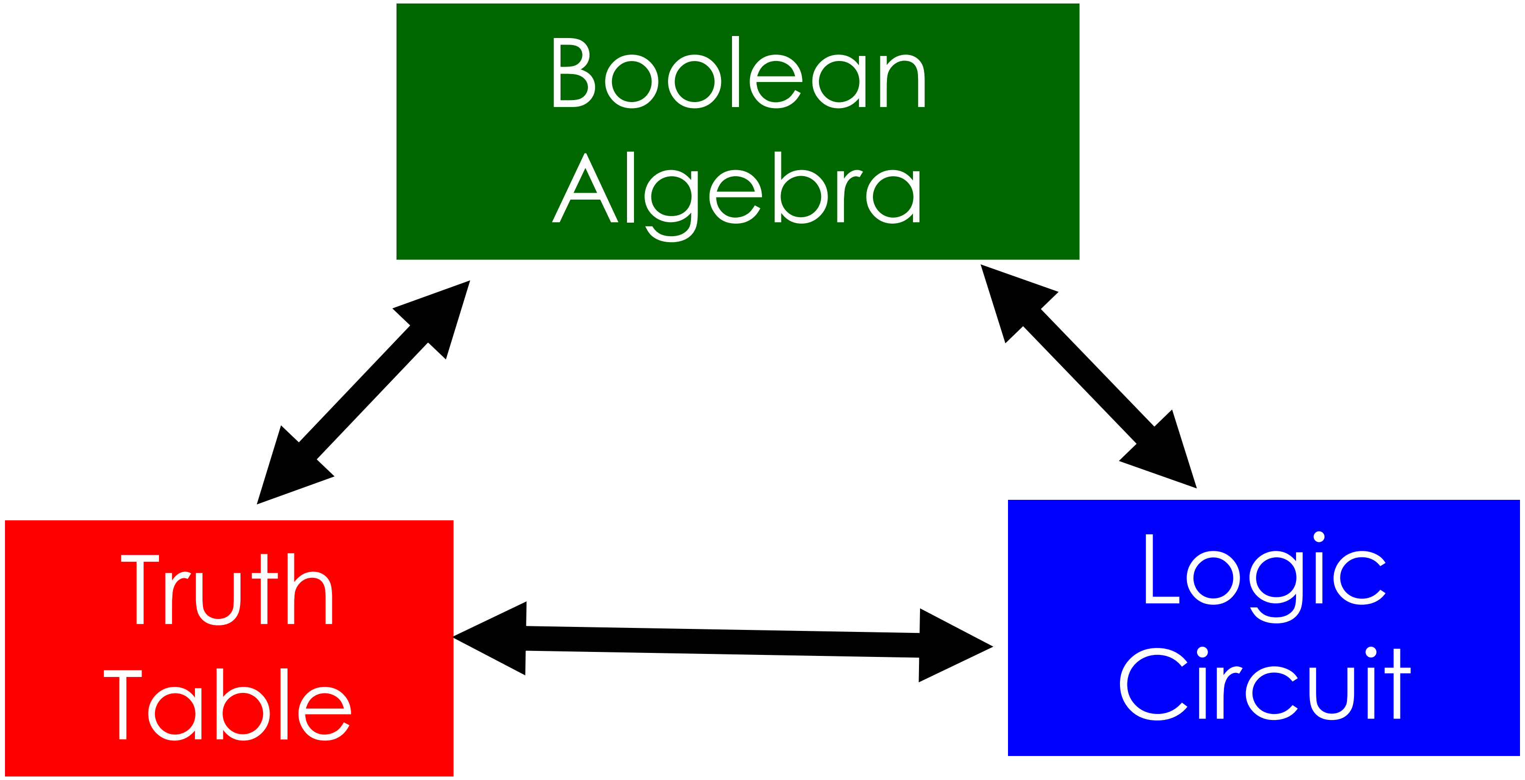
NOT X

X	$\overline{X}$
0	1
1	0

- A Boolean function has:
  - At least one Boolean variable,
  - At least one Boolean operator, and
  - At least one input from the set {0,1}
- It produces an output that is also a member of the set {0,1}

Now you know why the binary numbering system is so handy in digital systems

Conceptually



- Digital computers contain circuits that implement Boolean functions.
- The **simpler** that we can make a Boolean function, the **smaller** the circuit that will result.
  - Simpler circuits are cheaper to build, consume less power, and run faster than complex circuits.
- With this in mind, we always want to reduce our Boolean functions to their simplest form.
- There are a number of Boolean identities that help us to do this.

Properties of Boolean Algebra

- Most Boolean identities have an AND (product) form as well as an OR (sum) form.

Identity Name	AND Form	OR Form
Identity Law	$1x = x$	$0 + x = x$
Null Law	$0x = 0$	$1 + x = 1$
Idempotent Law	$xx = x$	$x + x = x$
Inverse Law	$x\overline{x} = 0$	$x + \overline{x} = 1$

- Our second group of Boolean identities should be familiar to you from your study of algebra:

Identity Name	AND Form	OR Form
Commutative Law	xy = yx	x+y = y+x
Associative Law	(xy)z = x(yz)	(x+y)+z = x + (y+z)
Distributive Law	x+yz = (x+y)(x+z)	x(y+z) = xy+xz

- Our last group of Boolean identities are perhaps the most useful.
- If you have studied set theory or formal logic, these laws are also familiar to you.

Identity Name	AND Form	OR Form
Absorption Law	$x(x+y) = x$	$x + xy = x$
DeMorgan's Law	$\overline{(xy)} = \bar{x} + \bar{y}$	$\overline{(x+y)} = \bar{x}\bar{y}$
Double Complement Law	$\overline{(\bar{x})} = x$	

- We can use Boolean identities to simplify the function:

$$F(X,Y,Z) = (X + Y)(X + \overline{Y})(\overline{X\overline{Z}})$$

as follows:

$(X + Y)(X + \overline{Y})(\overline{X\overline{Z}})$	Idempotent Law (Rewriting)
$(X + Y)(X + \overline{Y})(\overline{X} + Z)$	DeMorgan's Law
$(XX + X\overline{Y} + XY + Y\overline{Y})(\overline{X} + Z)$	Distributive Law
$((X + Y\overline{Y}) + X(Y + \overline{Y}))(\overline{X} + Z)$	Commutative & Distributive Laws
$((X + 0) + X(1))(\overline{X} + Z)$	Inverse Law
$X(\overline{X} + Z)$	Idempotent Law
$X\overline{X} + XZ$	Distributive Law
$0 + XZ$	Inverse Law
XZ	Idempotent Law

With respect to duality, Identities 1 – 8 have the following relationship:

1. $X + 0 = X$	2. $X \bullet 1 = X$	(dual of 1)
3. $X + 1 = 1$	4. $X \bullet 0 = 0$	(dual of 3)
5. $X + X = X$	6. $X \bullet X = X$	(dual of 5)
7. $X + X' = 1$	8. $X \bullet X' = 0$	(dual of 8)

Algebraic Manipulation

- **Boolean algebra is a useful tool for simplifying digital circuits.**
- **Why do it? Simpler can mean cheaper, smaller, faster.**
- **Example: Simplify $F = x'yz + x'yz' + xz$.**
  $$F = x'yz + x'yz' + xz$$
  $$= x'y(z+z') + xz$$
  $$= x'y \bullet 1 + xz$$
  $$= x'y + xz$$
- **Example: Prove $x'y'z' + x'yz' + xyz' = x'z' + yz'$**
- **Proof: $x'y'z' + x'yz' + xyz'$**
  $$= x'y'z' + x'yz' + x'yz' + xyz'$$
  $$= x'z'(y'+y) + yz'(x'+x)$$
  $$= x'z' \bullet 1 + yz' \bullet 1$$
  $$= x'z' + yz'$$

Complementation

- Sometimes it is more economical to build a circuit using the complement of a function (and complementing its result) than it is to implement the function directly.
- DeMorgan's law provides an easy way of finding the complement of a Boolean function.
- DeMorgan's law states:

$$\overline{(xy)} = \overline{x} + \overline{y} \quad \text{and} \quad \overline{(x+y)} = \overline{x}\,\overline{y}$$

- Find the complement of $F(x, y, z) = x\,y'\,z' + x'\,y\,z$

- $G = F' = (xy'z' + x'yz)'$

  $= (xy'z')' \bullet (x'yz)'$ *DeMorgan*

  $= (x'+y+z) \bullet (x+y'+z')$ *DeMorgan* again

- Note: The complement of a function can also be derived by finding the function's *dual*, and then complementing all of the literals

Truth Table

- Enumerates all possible combinations of variable values and the corresponding function value
- Truth tables for some arbitrary functions $F_1(x,y,z)$, $F_2(x,y,z)$, and $F_3(x,y,z)$ are shown to the right.
- Truth table: a <u>unique</u> representation of a Boolean function
- If two functions have identical truth tables, the functions are equivalent (and vice-versa).
- Truth tables can be used to prove equality theorems.
- However, the size of a truth table grows <u>exponentially</u> with the number of variables involved. This motivates the use of Boolean Algebra.

x	y	z		F_1	F_2	F_3
0	0	0		0	1	1
0	0	1		0	0	1
0	1	0		0	0	1
0	1	1		0	1	1
1	0	0		0	1	0
1	0	1		0	1	0
1	1	0		0	0	0
1	1	1		1	0	1

Standard SOP and POS

Overview

- Introduction
- SOP and POS
- Minterms and Maxterms
- Canonical Forms
- Conversion Between Canonical Forms
- Standard Forms

Introduction

- Through our exercises in simplifying Boolean expressions, we see that there are numerous ways of stating the same Boolean expression.
  - These “synonymous” forms are *logically equivalent*.
  - Logically equivalent expressions have identical truth tables.
- In order to eliminate as much confusion as possible, designers express Boolean functions in *standardized* or *canonical* form.

SOP and POS

- There are two canonical forms for Boolean expressions: Sum-Of-Products (SOP) and Product-Of-Sums (POS).
  - Recall the Boolean product is the AND operation and the Boolean sum is the OR operation.
- In the Sum-Of-Products form, ANDed variables are ORed together.
  - For example: $F(x,y,z) = xy + xz + yz$
- In the Product-Of-Sums form, ORed variables are ANDed together:
  - For example: $F(x,y,z) = (x+y)(x+z)(y+z)$

Definitions

- *Literal:* A variable or its complement
- *Product term:* literals connected by •
- *Sum term:* literals connected by +
- *Minterm:* a product term in which all the variables appear exactly once, either complemented or un-complemented
- *Maxterm:* a sum term in which all the variables appear exactly once, either complemented or un-complemented

Truth Table notation for Minterms and Maxterms

- Minterms and Maxterms are easy to denote using a truth table.
- Example:
  Assume 3 variables x,y,z (order is fixed)
- Any Boolean function F( ) can be expressed as a *unique* **sum** of **min**terms and a unique **product** of **max**terms (under a fixed variable ordering).
- In other words, every function F() has two canonical forms:
  - Canonical Sum-Of-Products (sum of minterms)
  - Canonical Product-Of-Sums (product of maxterms)

x	y	z		Minterm	Maxterm
0	0	0		$x'y'z' = m_0$	$x+y+z = M_0$
0	0	1		$x'y'z = m_1$	$x+y+z' = M_1$
0	1	0		$x'yz' = m_2$	$x+y'+z = M_2$
0	1	1		$x'yz = m_3$	$x+y'+z' = M_3$
1	0	0		$xy'z' = m_4$	$x'+y+z = M_4$
1	0	1		$xy'z = m_5$	$x'+y+z' = M_5$
1	1	0		$xyz' = m_6$	$x'+y'+z = M_6$
1	1	1		$xyz = m_7$	$x'+y'+z' = M_7$

Canonical Forms

- Canonical Sum-Of-Products:
  The minterms included are those m_j such that F( ) = 1 in row *j* of the truth table for F( ).
- Canonical Product-Of-Sums:
  The maxterms included are those M_j such that F( ) = 0 in row *j* of the truth table for F( ).

- $f_1(a,b,c) = \sum m(1,2,4,6)$, where $\sum$ indicates that this is a sum-of-products form, and $m(1,2,4,6)$ indicates that the minterms to be included are m_1, m_2, m_4, and m_6.
- $f_1(a,b,c) = \prod M(0,3,5,7)$, where $\prod$ indicates that this is a product-of-sums form, and $M(0,3,5,7)$ indicates that the maxterms to be included are M_0, M_3, M_5, and M_7.
- Since $m_j = M_j'$ for any *j*,
  $\sum m(1,2,4,6) = \prod M(0,3,5,7) = f_1(a,b,c)$

Conversion Between Canonical Forms

- Replace $\sum$ with $\prod$ (or *vice versa*) and replace those *j*'s that appeared in the original form with those that do not.
- Example:

$$f_1(a,b,c) = a'b'c + a'bc' + ab'c' + abc'$$
$$= m_1 + m_2 + m_4 + m_6$$
$$= \sum(1,2,4,6)$$
$$= \prod(0,3,5,7)$$
$$= (a+b+c)\bullet(a+b'+c')\bullet(a'+b+c')\bullet(a'+b'+c')$$

$$F = \overline{X}\,\overline{Y}\,\overline{Z} + \overline{X}Y\overline{Z} + X\overline{Y}Z + XYZ = m_0 + m_2 + m_5 + m_7 = \sum m(0, 2, 5, 7)$$

$$\overline{F} = \overline{X}\,\overline{Y}Z + \overline{X}YZ + X\overline{Y}\,\overline{Z} + XY\overline{Z} = m_1 + m_3 + m_4 + m_6 = \sum m(1, 3, 4, 6)$$

$$\overline{F} = m_1 + m_3 + m_4 + m_6$$

$$\Rightarrow F = \overline{m_1 + m_3 + m_4 + m_6} = \overline{m_1} \cdot \overline{m_3} \cdot \overline{m_4} \cdot \overline{m_6}$$

$$\Rightarrow F = M_1 \cdot M_3 \cdot M_4 \cdot M_6 = (X + Y + \overline{Z})(X + \overline{Y} + \overline{Z})(\overline{X} + Y + Z)(\overline{X} + \vec{Y} + Z)$$

$$= \prod M(1, 3, 4, 6)$$

Standard Forms

- Standard forms are "*like*" canonical forms, except that not all variables need appear in the individual product (SOP) or sum (POS) terms.
- Example:
  $f_1(a,b,c) = a'b'c + bc' + ac'$
  is a *standard* sum-of-products form
- $f_1(a,b,c) = (a+b+c)\bullet(b'+c')\bullet(a'+c')$
  is a *standard* product-of-sums form.

Conversion of SOP from standard to canonical form

- Expand *non-canonical* terms by inserting equivalent of 1 in each missing variable x:
  $(x + x') = 1$
- Remove duplicate minterms
- $f_1(a,b,c) = a'b'c + bc' + ac'$
  $= a'b'c + (a+a')bc' + a(b+b')c'$
  $= a'b'c + abc' + a'bc' + abc' + ab'c'$
  $= a'b'c + abc' + a'bc' + ab'c'$

Conversion of POS from standard to canonical form

- Expand non-canonical terms by adding 0 in terms of missing variables (*e.g.*, xx' = 0) and using the distributive law
- Remove duplicate maxterms
- $f_1(a,b,c) = (a+b+c)\bullet(b'+c')\bullet(a'+c')$

$$= (a+b+c)\bullet(aa'+b'+c')\bullet(a'+bb'+c')$$
$$= (a+b+c)\bullet(a+b'+c')\bullet(a'+b'+c')\bullet(a'+b+c')\bullet(a'+b'+c')$$
$$= (a+b+c)\bullet(a+b'+c')\bullet(a'+b'+c')\bullet(a'+b+c')$$

Minimization Techniques

Overview

- Introduction
- Karnaugh Map (K-Map)
- Simplification Rules
- K-Map Simplification for Two Variables
- K-Map Simplification for Three Variables
- K-Map Simplification for Four Variables
- Don’t Care Conditions
- Redundancy
- Design of Combinational Circuits

Introduction

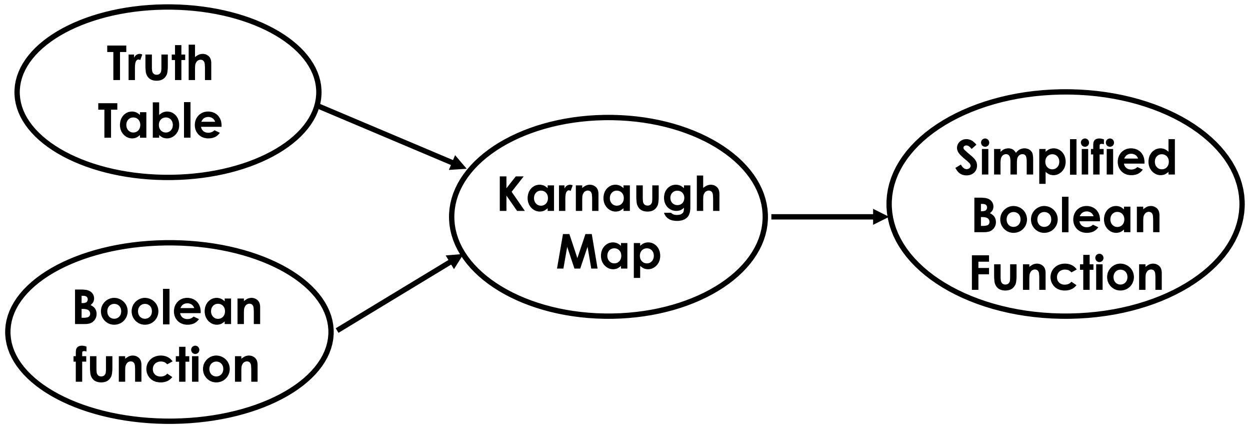
Truth Table → Boolean Function

Unique — Many different expressions exist

Simplification from Boolean function

- Finding an equivalent expression that is least expensive to implement
- For a simple function, it is possible to obtain a simple expression for low cost implementation
- But, with complex functions, it is a very difficult for implementation

Karnaugh Map (K-map) is a simple procedure for simplification of Boolean expressions.



Karnaugh Map (K-Map)

- Karnaugh maps (K-maps) are *graphical* representations of Boolean functions.
- One ***map cell*** corresponds to a row in the truth table.
- Also, one map cell corresponds to a minterm or a maxterm in the Boolean expression
- Each term is identified by a decimal number whose binary representation is identical to the binary interpretation of the input values of the term.

	C'D'	C'D	CD	CD'
A'B'	0	1	3	2
A'B	4	5	7	6
AB	12	13	15	14
AB'	8	9	11	10

K-Map Simplification for Two Variables

- Of course, the Minterm function that we derived from our Truth Table was not in simplest terms.
  - That's what we started with in this example.
- We can, however, reduce our complicated expression to its simplest terms by finding adjacent 1s in the K-map that can be collected into groups that are powers of two.

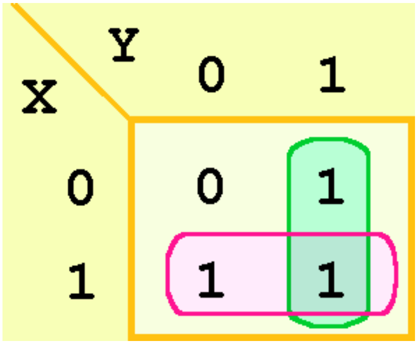
X \ Y	0	1
0	0	1
1	1	1

- In our example, we have two such groups.
  - Can you find them?

K-Map Rules

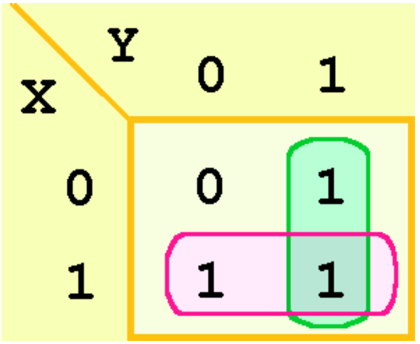
The rules of K-map simplification are:

- Groupings can contain only 1s; no 0s.
- **The number of 1s in a group must be a power of 2 – even if it contains a single 1.**
- Nearby 1s are to be grouped.
- Corner 1s are to be grouped.
- Group that wraps around the sides of a K-map.
- Diagonal groups are not allowed.
- The groups must be made as large as possible.
- Groups can overlap.



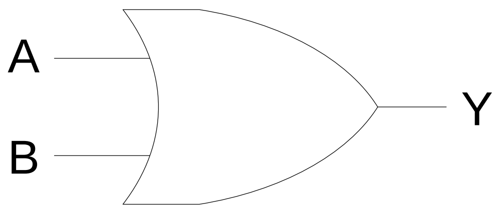
K-Map Rules

- The best way of selecting two groups of 1s form our simple K-map is shown.
- We see that both groups are powers of two and that the groups overlap.

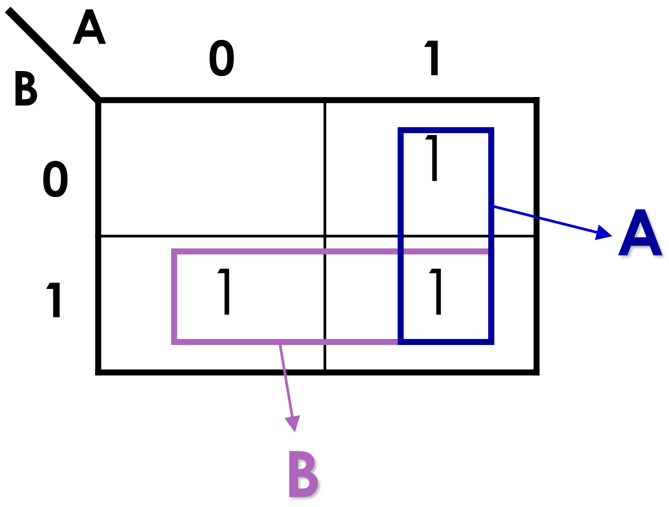


K-Map Simplification for Two Variables

2-variable Karnaugh maps are trivial but can be used to introduce the methods you need to learn. The map for a 2-input OR gate looks like this:



A	B	Y
0	0	0
0	1	1
1	0	1
1	1	1



A+B

K-Map Simplification for Three Variables

- A K-map for three variables is constructed as shown in the diagram below.
- We have placed each Minterm in the cell that will hold its value.
  - Notice that the values for the *yz* combination at the top of the matrix form a pattern that is not a normal binary sequence.

x \ yz	00	01	11	10
0	$\overline{X}\overline{Y}\overline{Z}$	$\overline{X}\overline{Y}Z$	$\overline{X}YZ$	$\overline{X}Y\overline{Z}$
1	$X\overline{Y}\overline{Z}$	$X\overline{Y}Z$	XYZ	$XY\overline{Z}$

- Consider the function:

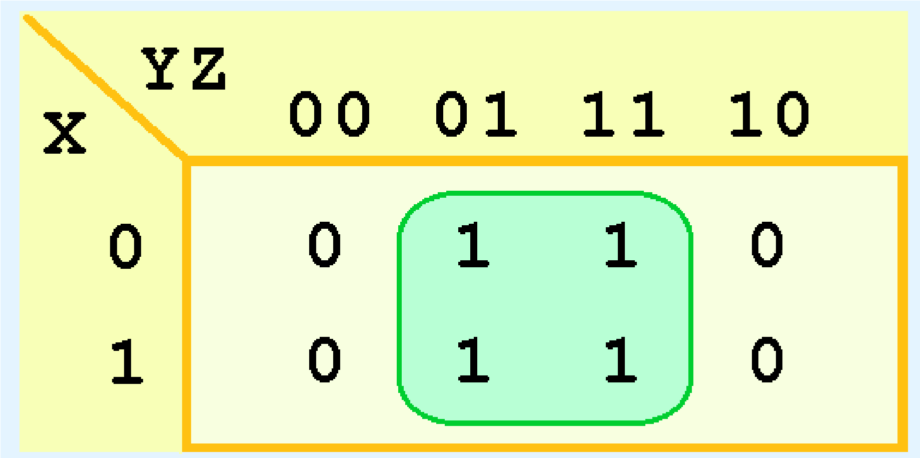
  F (X, Y, Z) = X'Y'Z + X'YZ + XY'Z + XYZ

- Its K-map is given below.
  - What is the largest group of 1s that is a power of 2?

X \ YZ	00	01	11	10
0	0	1	1	0
1	0	1	1	0

- This grouping tells us that changes in the variables *x* and *y* have no influence upon the value of the function: They are irrelevant.
- This means that the function, **F (X, Y, Z) = X'Y'Z + X'YZ + XY'Z + XYZ** reduces to ***F* = *Z***.

You could verify this reduction with Boolean Algebra



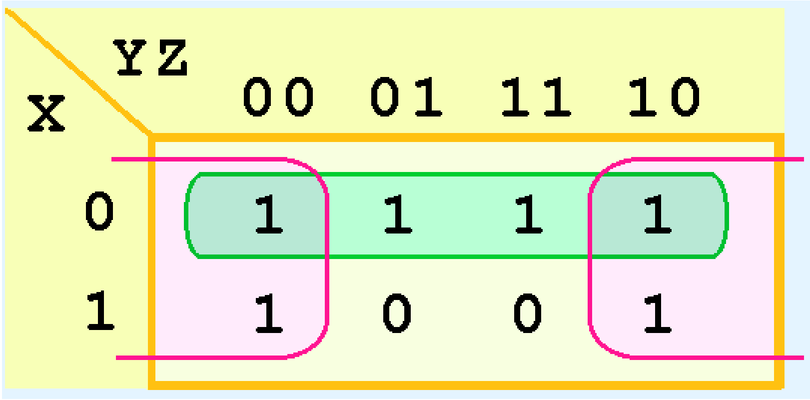
- Now for a more complicated K-map. Consider the function:

$$F(X,Y,Z) = \overline{X}\,\overline{Y}\,\overline{Z} + \overline{X}\,\overline{Y}Z + \overline{X}YZ + \overline{X}Y\overline{Z} + X\overline{Y}\,\overline{Z} + XY\overline{Z}$$

- Its K-map is shown below. There are (only) two groupings of 1s.
  - Can you find them?

X \ YZ	00	01	11	10
0	1	1	1	1
1	1	0	0	1

- In this K-map, we see an example of a group that wraps around the sides of a K-map.



$f = \sum(0,4) = \overline{B}\,\overline{C}$

A \ BC	00	01	11	10
0	1	0	0	0
1	1	0	0	0

$f = \sum(4,5) = A\,\overline{B}$

A \ BC	00	01	11	10
0	0	0	0	0
1	1	1	0	0

$f = \sum(0,1,4,5) = \overline{B}$

A \ BC	00	01	11	10
0	1	1	0	0
1	1	1	0	0

$f = \sum(0,1,2,3) = \overline{A}$

A \ BC	00	01	11	10
0	1	1	1	1
1	0	0	0	0

f = ∑ (1,3) = A'C

$\cancel{f = \sum(0,4) = \overline{A}\,C}$

A \ BC	00	01	11	10
0	0	1	1	0
1	0	0	0	0

$f = \sum(4,6) = A\,\overline{C}$

A \ BC	00	01	11	10
0	0	0	0	0
1	1	0	0	1

$f = \sum(0,2) = \overline{A}\,\overline{C}$

A \ BC	00	01	11	10
0	1	0	0	1
1	0	0	0	0

$f = \sum(0,2,4,6) = \overline{C}$

A \ BC	00	01	11	10
0	1	0	0	1
1	1	0	0	1

K-Map Simplification for Four Variables

- The K-map can be extended to accommodate the 16 Minterms that are produced by a four-input function.
- This is the format for a 16-minterm K-map.

WX \ YZ	00	01	11	10
00	$\overline{W}\,\overline{X}\,\overline{Y}\,\overline{Z}$	$\overline{W}\,\overline{X}\,\overline{Y}Z$	$\overline{W}\,\overline{X}YZ$	$\overline{W}\,\overline{X}Y\overline{Z}$
01	$\overline{W}X\overline{Y}\,\overline{Z}$	$\overline{W}X\overline{Y}Z$	$\overline{W}XYZ$	$\overline{W}XY\overline{Z}$
11	$WX\overline{Y}\,\overline{Z}$	$WX\overline{Y}Z$	$WXYZ$	$WXY\overline{Z}$
10	$W\overline{X}\,\overline{Y}\,\overline{Z}$	$W\overline{X}\,\overline{Y}Z$	$W\overline{X}YZ$	$W\overline{X}Y\overline{Z}$

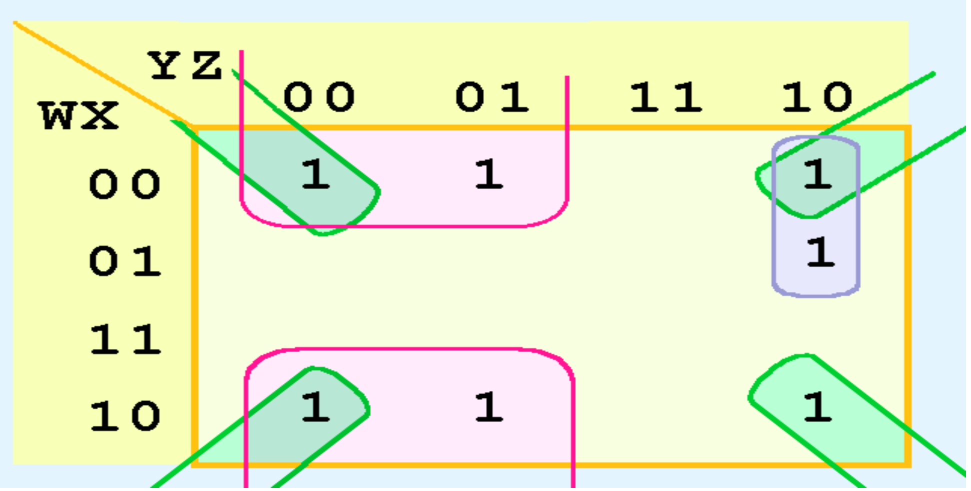
- We have populated the K-map shown below with the nonzero minterms from the function:

$$F(W,X,Y,Z)=\bar{W}\bar{X}\bar{Y}\bar{Z}+\bar{W}\bar{X}\bar{Y}Z+\bar{W}\bar{X}Y\bar{Z}$$
$$+\bar{W}XY\bar{Z}+W\bar{X}\bar{Y}\bar{Z}+W\bar{X}\bar{Y}Z+W\bar{X}Y\bar{Z}$$

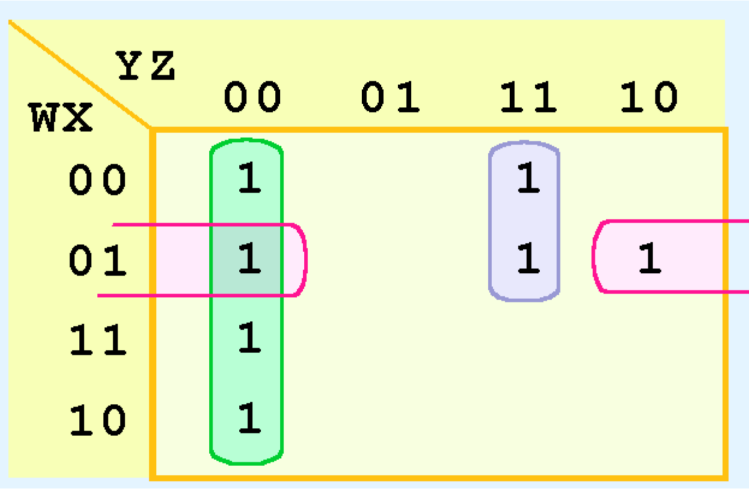
  - Can you identify (only) three groups in this K-map?

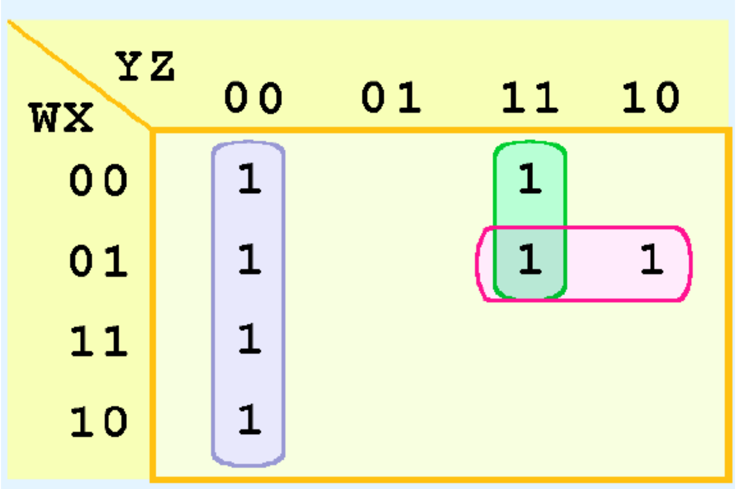
WX \ YZ	00	01	11	10
00	1	1		1
01				1
11				
10	1	1		1

- Our three groups consist of:
  - A purple group entirely within the K-map at the right.
  - A pink group that wraps the top and bottom.
  - A green group that spans the corners.
- Thus we have three terms in our final function:



- It is possible to have a choice as to how to pick groups within a K-map, while keeping the groups as large as possible.
- The (different) functions that result from the groupings below are logically equivalent.





AB \ CD	00	01	11	10
00	1	0	0	0
01	0	0	0	0
11	0	0	0	0
10	1	0	0	0

$$f = \sum(0,8) = \overline{B}\bullet\overline{C}\bullet\overline{D}$$

AB \ CD	00	01	11	10
00	0	0	0	0
01	0	1	0	0
11	0	1	0	0
10	0	0	0	0

$$f = \sum(5,13) = B\bullet\overline{C}\bullet D$$

AB \ CD	00	01	11	10
00	0	0	0	0
01	0	0	0	0
11	0	1	1	0
10	0	0	0	0

$$f = \sum(13,15) = A\bullet B\bullet D$$

AB \ CD	00	01	11	10
00	0	0	0	0
01	1	0	0	1
11	0	0	0	0
10	0	0	0	0

$$f = \sum(4,6) = \overline{A}\bullet B\bullet\overline{D}$$

AB \ CD	00	01	11	10
00	0	0	1	1
01	0	0	1	1
11	0	0	0	0
10	0	0	0	0

$$f = \sum(2,3,6,7) = \overline{A}\bullet C$$

AB \ CD	00	01	11	10
00	0	0	0	0
01	1	0	0	1
11	1	0	0	1
10	0	0	0	0

$$f = \sum(4,6,12,14) = B\bullet\overline{D}$$

AB \ CD	00	01	11	10
00	0	0	1	1
01	0	0	0	0
11	0	0	0	0
10	0	0	1	1

$$f = \sum(2,3,10,11) = \overline{B}\bullet C$$

AB \ CD	00	01	11	10
00	1	0	0	1
01	0	0	0	0
11	0	0	0	0
10	1	0	0	1

$$f = \sum(0,2,8,10) = \overline{B}\bullet\overline{D}$$

AB \ CD	00	01	11	10
00	0	0	0	0
01	1	1	1	1
11	0	0	0	0
10	0	0	0	0

$f = \sum(4,5,6,7) = \overline{A} \bullet B$

AB \ CD	00	01	11	10
00	0	0	1	0
01	0	0	1	0
11	0	0	1	0
10	0	0	1	0

$f = \sum(3,7,11,15) = C \bullet D$

AB \ CD	00	01	11	10
00	1	0	1	0
01	0	1	0	1
11	1	0	1	0
10	0	1	0	1

$f = \sum(0,3,5,6,9,10,12,15)$

$f = A \otimes B \otimes C \otimes D$

AB \ CD	00	01	11	10
00	0	1	0	1
01	1	0	1	0
11	0	1	0	1
10	1	0	1	0

$f = \sum(1,2,4,7,8,11,13,14)$

$f = A \oplus B \oplus C \oplus D$

AB \ CD	00	01	11	10
00	0	1	1	0
01	0	1	1	0
11	0	1	1	0
10	0	1	1	0

$f = \sum(1,3,5,7,9,11,13,15)$

$f = D$

AB \ CD	00	01	11	10
00	1	0	0	1
01	1	0	0	1
11	1	0	0	1
10	1	0	0	1

$f = \sum(0,2,4,6,8,10,12,14)$

$f = \overline{D}$

AB \ CD	00	01	11	10
00	0	0	0	0
01	1	1	1	1
11	1	1	1	1
10	0	0	0	0

$f = \sum(4,5,6,7,12,13,14,15)$

$f = B$

AB \ CD	00	01	11	10
00	1	1	1	1
01	0	0	0	0
11	0	0	0	0
10	1	1	1	1

$f = \sum(0,1,2,3,8,9,10,11)$

$f = \overline{B}$

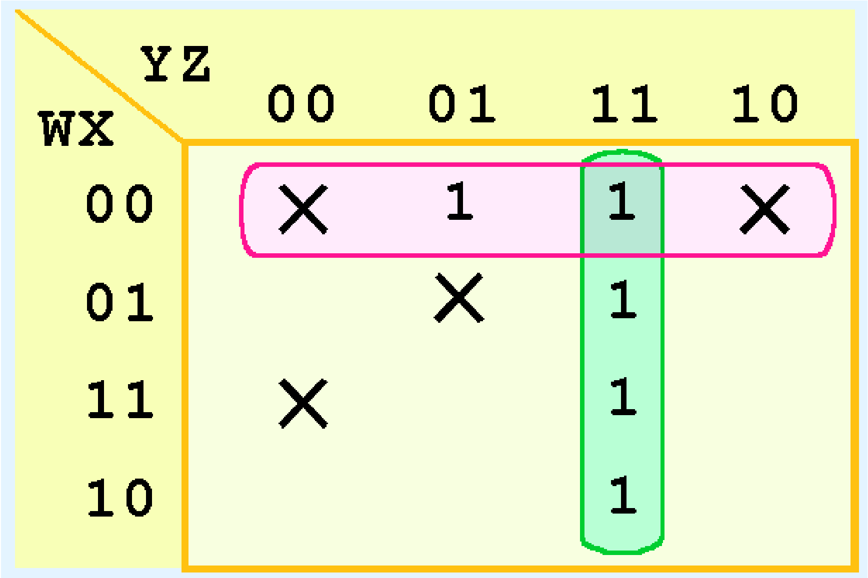
Don’t Care Conditions

- Real circuits don’t always need to have an output defined for every possible input.
  - For example, some calculator displays consist of 7-segment LEDs. These LEDs can display 2^7 patterns but all patterns are not used.
- If a circuit is designed so that a particular set of inputs can never happen, we call this set of inputs a *don’t care* condition.
- They are very helpful to us in K-map circuit simplification.

- In a K-map, a don't care condition is identified by an *X* in the cell of the minterm(s) for the don't care inputs, as shown below.
- In performing the simplification, we are free to include or ignore the *X*'s when creating our groups.

WX \ YZ	00	01	11	10
00	X	1	1	X
01		X	1	
11	X		1	
10			1	

- In one grouping in the K-map below, we have the function:
- **F = W’X’ + YZ**



- A different grouping gives us the function:

$$F(W,X,Y,Z) = \overline{W}Z + YZ$$

WX \ YZ	00	01	11	10
00	X	1	1	X
01		X	1	
11	X		1	
10			1	

- The truth table of:

$$F(W, X, Y, Z) = W'X' + YZ$$

differs from the truth table of:

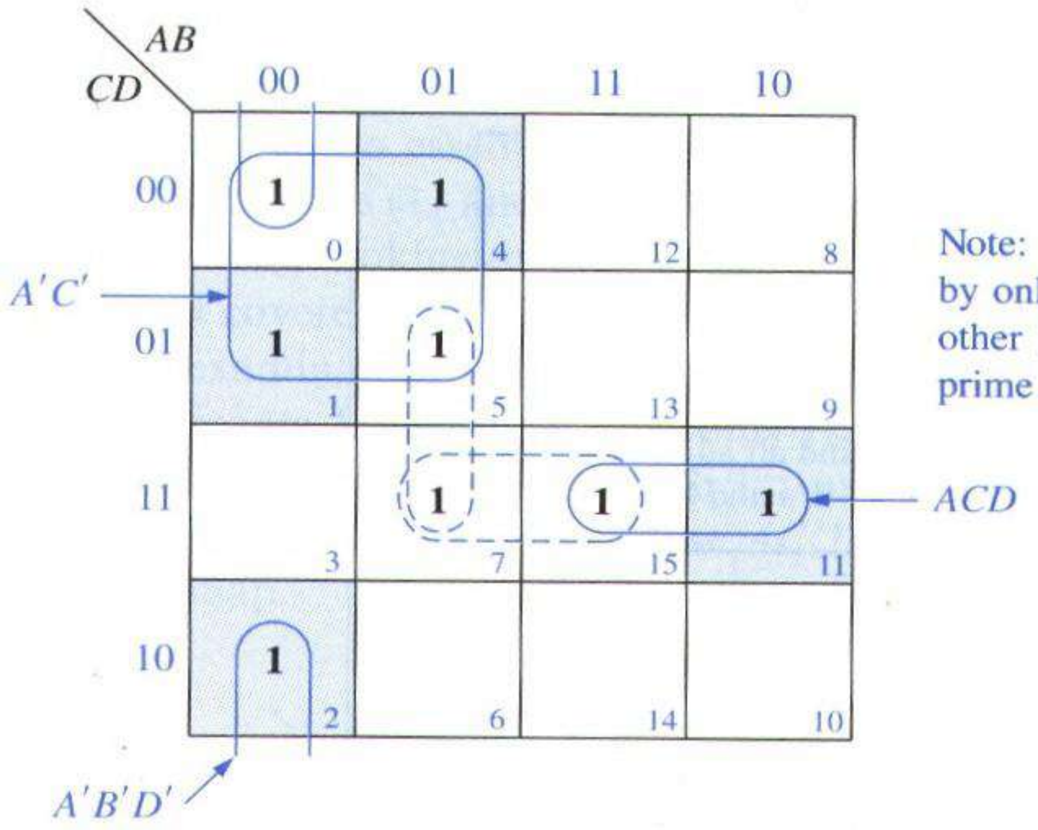
$$F(W,X,Y,Z) = \overline{W}Z + YZ$$

- However, the values for which they differ, are the inputs for which we have don't care conditions.

WX \ YZ	00	01	11	10
00	X	1	1	X
01		X	1	
11	X		1	
10			1	

WX \ YZ	00	01	11	10
00	X	1	1	X
01		X	1	
11	X		1	
10			1	

Redundancy



Note: 1's shaded in blue are covered by only one prime implicant. All other 1's are covered by at least two prime implicants.

Design of combinational digital circuits

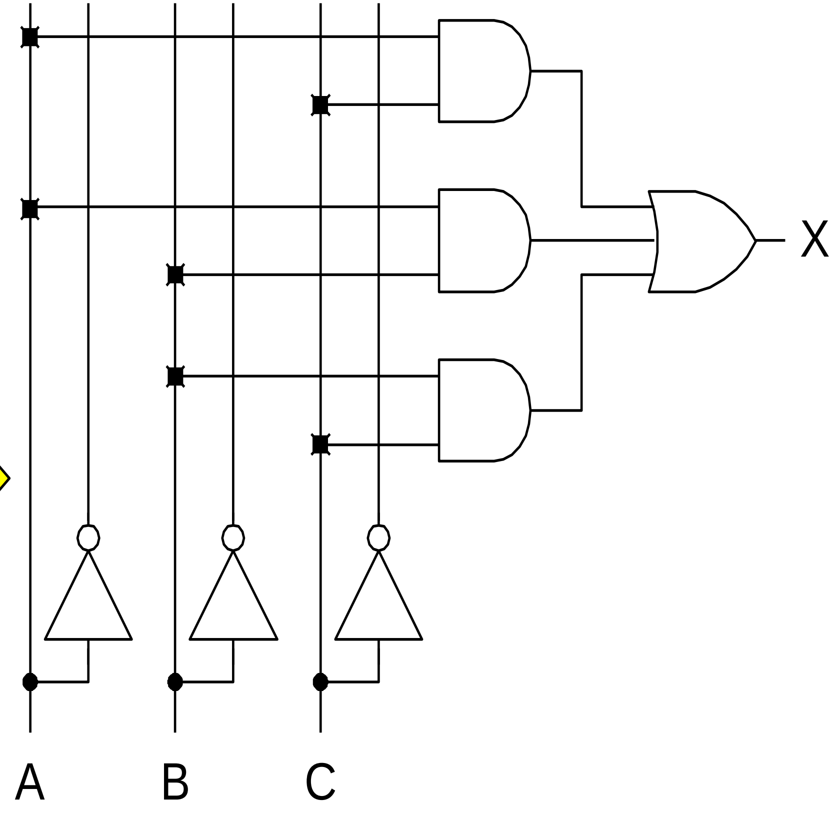
- Steps to design a combinational digital circuit:
  - From the problem statement derive the truth table
  - From the truth table derive the unsimplified logic expression
  - Simplify the logic expression
  - From the simplified expression draw the logic circuit
- Example: Design a 3-input (A,B,C) digital circuit that will give at its output (X) a logic 1 only if the binary number formed at the input has more ones than zeros.

	Inputs			Output
	A	B	C	X
0	0	0	0	0
1	0	0	1	0
2	0	1	0	0
3	0	1	1	1
4	1	0	0	0
5	1	0	1	1
6	1	1	0	1
7	1	1	1	1

$$X = \sum(3, 5, 6, 7)$$

A \ BC	00	01	11	10
0	0	0	1	0
1	0	1	1	1

$$X = AC + AB + BC$$



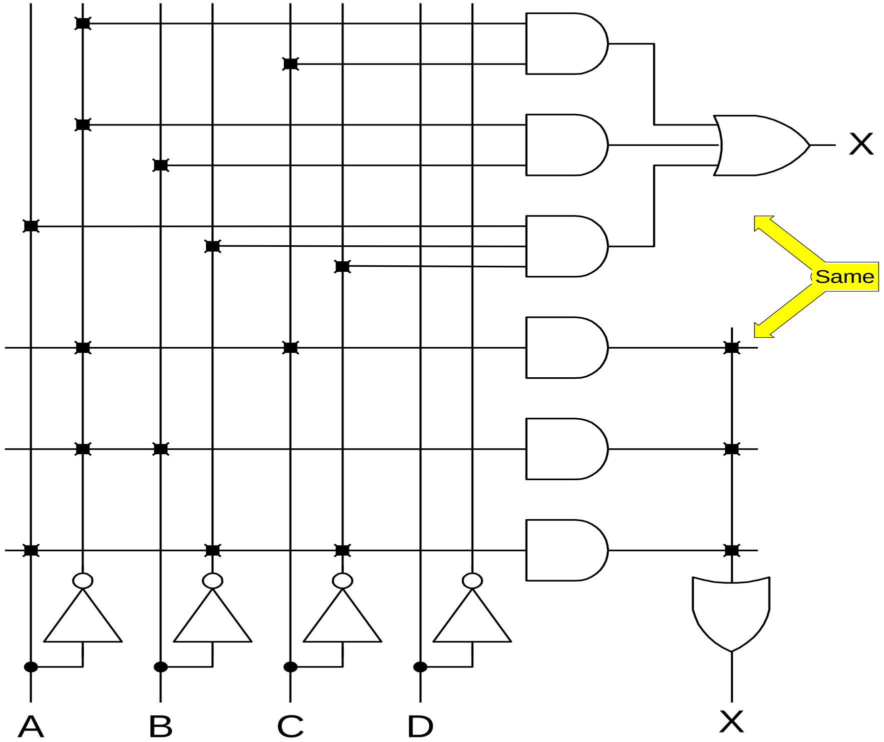
- Example: Design a 4-input (A,B,C,D) digital circuit that will give at its output (X) a logic 1 only if the binary number formed at the input is between 2 and 9 (including).

	Inputs A	B	C	D	Output X
0	0	0	0	0	0
1	0	0	0	1	0
2	0	0	1	0	1
3	0	0	1	1	1
4	0	1	0	0	1
5	0	1	0	1	1
6	0	1	1	0	1
7	0	1	1	1	1
8	1	0	0	0	1
9	1	0	0	1	1
10	1	0	1	0	0
11	1	0	1	1	0
12	1	1	0	0	0
13	1	1	0	1	0
14	1	1	1	0	0
15	1	1	1	1	0

$$X = \sum(2,3,4,5,6,7,8,9)$$

AB \ CD	00	01	11	10
00	**0**	**0**	**1**	**1**
01	**1**	**1**	**1**	**1**
11	**0**	**0**	**0**	**0**
10	**1**	**1**	**0**	**0**

$$X = \overline{A}C + \overline{A}B + A\overline{B}\,\overline{C}$$



Conclusion

- K-maps provide an easy graphical method of simplifying Boolean expressions.
- A K-map is a matrix consisting of the outputs of the minterms of a Boolean function.
- In this section, we have discussed 2- 3- and 4-input K-maps. This method can be extended to any number of inputs through the use of multiple tables.

Recapping the rules of K-map simplification:

- Groupings can contain only 1s; no 0s.
- Groups can be formed only at right angles; diagonal groups are not allowed.
- The number of 1s in a group must be a power of 2 – even if it contains a single 1.
- The groups must be made as large as possible.
- Groups can overlap and wrap around the sides of the K-map.
- Use don’t care conditions when you can.
- Redundancy must be reduced