Eigen functions Eigen Values & Corresponding of the Integral Equation

$$y(x) = \lambda \int_0^1 (2xt - 4x^2)\, y(t)\, dt$$

Sol →

$$y(x) = \lambda \int_0^1 (2xt - 4x^2)\, y(t)\, dt$$

$$y(x) = 2\lambda x \int_0^1 t\, y(t)\, dt - 4\lambda x^2 \int_0^1 y(t)\, dt \quad \text{(1)}$$

$$c_1 = \int_0^1 t\, y(t)\, dt \rightarrow (2)$$

$$c_2 = \int_0^1 y(t)\, dt \rightarrow (3)$$

then (1) reduces to $y(x) = 2\lambda c_1 x - 4\lambda c_2 x^2 \rightarrow (4)$

$$y(t) = 2\lambda c_1 t - 4\lambda c_2 t^2 \quad (5)$$

Put the value of $y(t)$ in (2) & (3)

$$c_1 = \int_0^1 t\{2\lambda c_1 t - 4\lambda c_2 t^2\}\, dt \qquad c_2 = \int_0^1 2\lambda c_1 t - 4\lambda c_2 t^2\, dt$$

$$c_1 = 2\lambda c_1 \left[\frac{t^3}{3}\right]_0^1 - 4\lambda c_2 \left[\frac{t^4}{4}\right]_0^1 \qquad c_2 = 2\lambda c_1 \left(\frac{t^2}{2}\right)_0^1 - 4\lambda c_2 \left(\frac{t^3}{3}\right)_0^1$$

$$c_1 = \frac{2\lambda c_1}{3} - \frac{\cancel{4}\lambda c_2}{\cancel{4}} \qquad c_2 = \frac{\cancel{2}\lambda c_1}{\cancel{2}} - \frac{4\lambda c_2}{3}$$

$$\left(1 - \frac{2\lambda}{3}\right) c_1 + \lambda c_2 = 0 \quad (6) \qquad -c_2\left(1 + \frac{4\lambda}{3}\right) + \lambda c_1 = 0$$

$$\text{or } \lambda c_1 - c_2\left(1 + \frac{4\lambda}{3}\right) = 0 \quad (7)$$

Thus we have a system of linear Eq^n (6) & (7) for determining c_1 & c_2

For non-zero solution of this system of Equations

$$\begin{vmatrix} 1-\frac{2\lambda}{3} & \lambda \\ \lambda & -\left(1+\frac{4\lambda}{3}\right) \end{vmatrix} = 0$$

$$-\left(1-\frac{2\lambda}{3}\right)\left(1+\frac{4\lambda}{3}\right) - \lambda^2 = 0$$

$$\lambda^2 + 6\lambda + 9 = 0 \Rightarrow (\lambda+3)^2 = 0$$

$$\lambda = -3, -3$$

Hence the Eigen Values are $\lambda_1 = -3$, ~~3~~ ; $\lambda_2 = -3$

To determine Eigen function corresponding to $\lambda = \lambda_1 = 3$

Putting $\lambda = \lambda_1 = -3$ in (6) & (7)

$$c_1\left(1-\frac{2\lambda}{3}\right) + \lambda c_2 = 0$$

$$\lambda c_2 - c_2\left(1+\frac{4\lambda}{3}\right) = 0$$

$$3c_1 - 3c_2 = 0 \rightarrow (8)$$

$$-3c_1 + 3c_2 = 0 \rightarrow (9)$$

(8) or (9) given $c_1 = c_2$ Hence from (4)

$$y(x) = 2\lambda c_1 x - 4\lambda c_1 x^2$$

$$= 2\lambda c_1 (x - 2x^2) \quad \text{But } \lambda = -3$$

$$= -6c_1 (x - 2x^2)$$

taking $-6c_1 = 1$ the Eigen function is $x - 2x^2$

Hence Eigen function corresponding to Eigenvalue $\lambda_1 = \lambda_2 = -3$ is $x - 2x^2$