# Exact DIFFERENTIAL EQUATIONS

A first order differential equation of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0 \rightarrow (1)$$

is called an exact differential equation if it can be derived from its integral (i.e. solution) directly by differentiation without any multiplication or elimination. In other words, if there exists some function $u(x,y)$ such that $du = M\,dx + N\,dy$ then the equation is exact and its general solution is $u(x,y) = C$

**Theorem** → The necessary and sufficient condition for the differential equation $M\,dx + N\,dy = 0$ to be exact is $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

**Proof** → let us first show that condition is necessary. Suppose the differential eq$^n$ is exact, then

$$M\,dx + N\,dy = du \rightarrow (1)$$ where $u$ is some function of $x$ and $y$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \rightarrow (2)$$

$$M\,dx + N\,dy = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

Equating the coefficients of $dx$ and $dy$ on Both side given $M = \frac{\partial u}{\partial x}$ ... $\frac{\partial u}{\partial y}$

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

$$\text{so} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \because \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

which means that the Condition is necessary.
Now let us take up the sufficient part of the theorem.

Given that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ we are to show the exactness part.

Let $\int M\,dx = u$ where $y$ is kept constant during the in...

$$\frac{\partial}{\partial x}\left[\int M\,dx\right] = \frac{\partial u}{\partial x} \Rightarrow M = \frac{\partial u}{\partial x}$$

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \Rightarrow \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial y \partial x}$$

$$N = \frac{\partial u}{\partial y} + f(y) \quad \text{const. of integration}$$

Because $M\,dx + N\,dy$

$$\frac{\partial u}{\partial x}\,dx + \left[\frac{\partial u}{\partial y} + f(y)\right] dy$$

$$\frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy + f(y)\,dy$$

$$= du + f(y)\,dy$$

$$= d\left[u + \int f(y)\,dy\right]$$

$M\,dx + N\,dy = 0$ is Exact

...eral Solution

Integrate M w.r.t x treating y as a constant, then integrate w.r.t y only those terms in N which do not involve x. The sum of the two expressions thus obtained equated to an arbitrary constant is the required solution." That is

$$\int (\text{all terms in } M)\,dx + \int (\text{only those terms in } N \text{ not involving } x)\,dy$$

or

$$\int (\text{all terms in } N)\,dy + \int (\text{only those terms in } M \text{ not involving } y)\,dx = C$$