Theorem → The expectation of the product of two independent random Variable is equal to the product of their expectations i.e

$$E(X.Y) = E(X).E(Y)$$

Proof → Let $p(x,y)$ be the joint probability function of X and Y. For independent Variables

$$p(x,y) = p_1(x)\, p_2(y) \longrightarrow (1)$$

(i) **Discrete Case**

$$E(X,Y) = \sum_{i}^{m} \sum_{j}^{n} (x_i y_j)\, P(x_i y_j)$$

$$= \sum\sum x_i y_j\, p_1(x_i)\, p_2(y_j)$$

$$= \sum_{i=1}^{m} x_i\, p_1(x_i) \sum_{j=1}^{n} y_j\, p_2(y_j)$$

$$= E(X).E(Y)$$

(ii) **Continuous Case** →

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, p(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, p_1(x)\, p_2(y)\, dx\, dy \quad \text{from } (1)$$

$$= \int_{-\infty}^{\infty} x\, p_1(x)\, dx \int_{-\infty}^{\infty} y\, p_2(y)\, dy$$

$$= E(X)\, E(Y)$$

In general → $E(A.B.C \ldots X.Y.Z) = E(A).E(B) \ldots E(Y).E(Z)$

Example → A number is chosen at random from the set 1, 2, 3, —— 100 and another number is chosen at random from the set 1, 2, —— 50. what is the expected value of the product

Solution ⇒ let x be the number chosen at random from the set 1, 2, 3 —— 100, then x is a discrete random variable whose probability distribution is given by

x	1	2	3	- - - -	100	Total
p	$\frac{1}{100}$	$\frac{1}{100}$	$\frac{1}{100}$	— ——	$\frac{1}{100}$	1

$$\therefore E(x) = \sum p_i x_i = \frac{1}{100}\times 1 + \frac{1}{100}\times 2 + \frac{1}{100}\times 3 + \text{——} + \frac{1}{100}\times 100$$

$$= \frac{1}{100}\left[1 + 2 + 3 + \text{——} 100\right]$$

Sum of 'n' natural numbers $\frac{n(n+1)}{2}$

$$= \frac{1}{100} \times \frac{100\,(100+1)}{2}$$

$$= \frac{1}{100} \times \frac{100 \times 101}{2} = \frac{101}{2}$$

If y be the number chosen at random from the set 1, 2, —— 50 then the probability distribution of y is given by

x	1	2	3	— — —	50	Total
p'	$\frac{1}{50}$	$\frac{1}{50}$	$\frac{1}{50}$	—— ——	$\frac{1}{50}$	1

$$E(y) = \sum p'_i y_i = \frac{1}{50}\times 1 + \frac{1}{50}\times 2 + \frac{1}{50}\times 3 + \text{——} \frac{1}{50}\times 50$$

$$= \frac{1}{50}\left[1 + 2 + 3 + \text{——} + 50\right] = \frac{1}{50} \cdot \frac{50 \times 50+1}{2} = \frac{51}{2}$$

Clearly x and y are two independent random variables

Hence expected value of the product $E(xy)$

$$E(xy) = E(x).\,E(y)$$

$$= \frac{101}{2} \cdot \frac{51}{2} = \frac{5151}{4}$$