Tholm First Fundmental Theorem

atement → If (a) $D(\lambda) \neq 0$

(b) $K(x, t)$ is continuous in R

(c) $f(x)$ is continuous in I

then the equation $u(x) = f(x) + \lambda \int_a^b K(x, t) u(t) dt$ has one & only one solution given by

$$u(x) = f(x) + \int_a^b \frac{D(x, t; \lambda) f(t)}{D(\lambda)} dt$$

where $D(x, t; \lambda)$ and $D(\lambda)$ are absolutely and permanently convergent series in λ and $D(x, t; \lambda)$ converges uniformly w.r.t x & t on R

$R: a \le x \le b, \quad a \le y \le b$

Proof → Consider the Fredholm integral Equation of second kind

$$u(x) = f(x) + \lambda \int_a^b K(x, t) u(t) dt \rightarrow ①$$

or $u(x) = f(x) + \lambda \int_a^b K(x, t_1)$ d

$$u(t) = f(t) + \lambda \int_a^b K(t, t_1) u(t_1) dt_1 \rightarrow ②$$

Multiply ② by $D(x, t; \lambda)$ on both side and Integrate from a to b, we get

$$\int^b D(x, t; \lambda) u(t) dt = \int_a^b D(x, t; \lambda) f(t) dt + \lambda \int_a^b \int_a^b D(x, t; \lambda$$
$$K(t, t_1)$$
$$dt$$

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$$\int_a^b D(x,t;\lambda)u(t)\,dt = \int_a^b D(x,t;\lambda)f(t)\,dt + \int_a^b u(t_1)\Big[\lambda\int_a \; dt\Big]_a$$

~~Rem~~ Now using Fredholm's relation

$$D(x,y;\lambda) - \lambda K(x,y)D(\lambda) = \lambda\int_a^b D(x,t;\lambda)K(t,y)\,dt$$

$$\Rightarrow \int_a^b D(x,t;\lambda)u(t)\,dt = \int_a^b D(x,t;\lambda)f(t)\,dt$$

$$+\int_a^b \cancel{D(x,t_1;\lambda)u(t_1)dt} - \quad \text{here } y\to t_1$$

$$+\int_a^b u(t_1)\big[D(x,t_1;\lambda) - \lambda K(x,t_1)D(\lambda)\big]\,dt$$

$$\Rightarrow \int_a^b \cancel{D(x,t;\lambda)u(t)dt} = \int_a^b D(x,t;\lambda)f(t)\,dt$$

$$+\int_a^b \cancel{D(x,t_1;\lambda)u(t_1)dt_1} - \lambda\int_a^b K(x,t_1)u(t_1)D\,dt$$

$$\Rightarrow \lambda\int_a^b K(x,t_1)u(t_1)\,dt_1 = \int_a^b \frac{D(x,t;\lambda)}{D(\lambda)}f(t)\,dt \;\; \hookrightarrow ⊛$$

Putting this result in Ⓟ or we get

$$u(x) = f(x) + \int_a^b \frac{D(x,t;\lambda)}{D(\lambda)}f(t)\,dt \;\rightarrow ③$$

Now, we shall prove that ③ is a solution of ①