(H)omogeneous Differential Equation →

differential Equation of the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ is called a homogeneous equation where $f(x,y)$ and $g(x,y)$ are homogeneous functions in x and y of the same degree.

Ex → $$\frac{dy}{dx} = \frac{3xy + y^2}{3x^2 + xy}$$ (same degree)

Such differential equations can be solved by putting $y = vx$ so that the dependent variable y is changed to another variable v and then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ and the equation reduces to the variable separable form

Example → Solve $y^2 dx + (x^2 - xy + y^2) dy = 0$

Solution → $\frac{dy}{dx} = \frac{-y^2}{x^2 - xy + y^2}$

This is of homogeneous form hence putting $y = vx$ gives

$$v + x\frac{dv}{dx} = \frac{-v^2x^2}{x^2 - x^2v + v^2x^2} = \frac{-v^2}{1 - v + v^2}$$

$$x\frac{dv}{dx} = \frac{-v^2}{1 - v + v^2} - v$$

$$x\frac{dv}{dx} = -\frac{v^3 - v}{1 - v + v^2}$$

$$\frac{1 - v + v^2}{-v^3 - v}\, dv = \frac{dx}{x}$$

$$\frac{1 - v + v^2}{-v(v^2 + 1)}\, dv = \frac{dx}{x}$$

$$\left(-\frac{1}{v} + \frac{1}{v^2+1}\right) dv = \frac{dx}{x} \Rightarrow \frac{dx}{x} = \left(\frac{-1}{v} + \frac{1}{v^2+1}\right) dv$$

on Integration

$$-\log v + \tan^{-1} v + \log c = \log x$$

Put $v = \frac{y}{x}$

$$\log x = -\log\left(\frac{y}{x}\right) + \tan^{-1}\left(\frac{y}{x}\right) + \log c$$

$$\cancel{\log x} = -\log y + \cancel{\log x} + \tan^{-1}\frac{y}{x} + \log c$$

$$\log y \; - \; \tan^{-1} y$$