Find all the possible Taylor's and Laurent series expansions of the function $f(z)$ about the point $z=1$ where $f(z) = \dfrac{1}{(z+1)(z+2)^2}$

Solution → $f(z) = \dfrac{1}{(z+1)(z+2)^2} = \dfrac{1}{z+1} - \dfrac{1}{z+2} - \dfrac{1}{(z+2)^2}$

$f(z)$ is not defined at $z=-1,-2$

Distance of $z=1$ from $z=-1,-2$ are respectively 2 & 3.

We shall have Taylor Expansion in $|z-1|<2$ & Laurent Expansion is $2<|z-1|<3$ and $|z-1|>3$

In the region $|z-1|<2$ $\qquad \left|\dfrac{z-1}{2}\right|<1$

$$f(z) = \frac{1}{z+1} - \frac{1}{z+2} - \frac{1}{(z+2)^2}$$

$$= \frac{1}{z-1+2} - \frac{1}{z-1+3} - \frac{1}{(z-1+3)^2}$$

$$= \frac{1}{2}\frac{1}{\left(1+\frac{z-1}{2}\right)} - \frac{1}{3}\frac{1}{\left(1+\frac{z-1}{3}\right)} - \frac{1}{3^2\left(1+\frac{z-1}{3}\right)^2}$$

$$= \frac{1}{2}\left(1+\frac{z-1}{2}\right)^{-1} - \frac{1}{3}\left(1+\frac{z-1}{3}\right)^{-1} - \frac{1}{9}\left(1+\frac{z-1}{3}\right)^{-2}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^n}{2^n} - \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^n}{3^n} - \frac{1}{3^2}\sum_{n=0}^{\infty}(-1)^n(n+1)\frac{(z-1)^n}{3^n}$$

$$= \sum_{n=0}^{\infty}(-1)^n\left[\frac{1}{2^{n+1}} - \frac{n+4}{3^{n+2}}\right](z-1)^n$$

$(1+x)^{-2} = 1-2x+3x^2-4x^3+ \ldots$

In the region $2<|z-1|<3$ $\qquad \frac{|z-1|}{3}<1$, $\frac{2}{|z-1|}<1$

$$f(z)=\frac{1}{z+1}-\frac{1}{z+2}-\frac{1}{(z+2)^2}$$

$$=\frac{1}{z-1+2}+\frac{1}{z-1+3}+\frac{1}{(z-1+3)^2}$$

$$=\frac{1}{z-1}\left[1+\frac{2}{z-1}\right]^{-1}+\frac{1}{3}\left(1+\frac{z-1}{3}\right)^{-1}-\frac{1}{3^2}\left(1+\frac{z-1}{3}\right)^{-2}$$

$$=\frac{1}{z-1}\sum_{n=0}^{\infty}(-1)^n\frac{2^n}{(z-1)^n}-\frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^n}{3^n}-\frac{1}{3^2}\sum_{n=0}^{\infty}(-1)^n(n+1)\frac{(z-1)^n}{3^n}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n 2^n}{(z-1)^{n+1}}-\sum_{n=0}^{\infty}\frac{(-1)^n(n+4)}{3^{n+1}}(z-1)^n$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{n-1}}{(z-1)^n}-\sum_{n=0}^{\infty}\frac{(-1)^n(n+4)}{3^{n+2}}(z-1)^n$$

Replace n by n-1

In the region $|z-1|>3$ $\qquad \frac{3}{z-1}<1$

$$f(z)=\frac{1}{z+1}-\frac{1}{z+2}-\frac{1}{(z+2)^2}=\frac{1}{z-1+2}-\frac{1}{z-1+3}-\frac{1}{(z-1+3)^2}$$

$$=\frac{1}{z-1}\left[1+\frac{2}{z-1}\right]^{-1}-\frac{1}{z-1}\left[1+\frac{3}{z-1}\right]^{-1}-\frac{1}{(z-1)^2}\left[1+\frac{3}{z-1}\right]$$

$$=\frac{1}{z-1}\sum_{n=0}^{\infty}\frac{(-1)^n 2^n}{(z-1)^n}-\frac{1}{z-1}\sum_{n=0}^{\infty}(-1)^n\frac{3^n}{(z-1)^n}-\frac{1}{(z-1)^2}\sum_{n=0}^{\infty}(-1)^n(n+1)\frac{3^n}{(z-1)^n}$$

$$\sum_{n=0}^{\infty}\frac{2^n(-1)^n}{(z-1)^{n+1}}-\sum_{n=0}^{\infty}\frac{(-1)^n 3^n}{(z-1)^{n+1}}-\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)3^n}{(z-1)^{n+2}}$$

$$\sum_{n=1}^{\infty}\frac{2^{n-1}(-1)^{n-1}}{(z-1)^n}-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}3^{n-1}}{(z-1)^n}-\sum_{n=2}^{\infty}\frac{(-1)^{n-2}(n-1)3^{n-2}}{(z-1)^n}$$

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Ans.