Linear Differential Equations

A differential equation of the form

$$\frac{dy}{dx} + Py = Q \rightarrow (1)$$

is called a linear Differential Equation where P and Q are functions of x (but not of y) or Constants.

If the right side Q is zero for all x in the interval in which we consider ($Q \equiv 0$) (that is $Q = 0$ for all x considered) or $Q(x)=0$ the equation is said to be homogeneous otherwise it is said to be non homogeneous

let us find a formula for the general solution of (1) in some Interval I

for the homogeneous Equation

$$\frac{dy}{dx} + Py = 0 \rightarrow (2)$$

By seperating variables

$$\frac{dy}{y} = -P\,dx$$

on Integration

$$\log y = -\int P\,dx + \log c$$

$$\log \frac{y}{c} = -\int P\,dx$$

By Taking exponential on Both sides

$$y = C e^{-\int P\,dx} \rightarrow (3)$$

here we may also take $C=0$ and obtain the trivial solution $y(x) \equiv 0$

The non homogeneous Equation ① wil

Eqn ① can be written as

$$\frac{dy}{dx} + Py = Q$$

$$\frac{dy}{dx} + (Py - Q) = 0 \Rightarrow (Py-$$

where $M = Py - Q$, $N = 1$

$$\frac{\partial M}{\partial y} = P \qquad \frac{\partial N}{\partial x} = 0$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Now $\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{1}{1}\left[P - 0\right] =$

so $I.F = e^{\int P dx}$

Now Multiply the given eqn by e

$$e^{\int P dx}\left[\frac{dy}{dx} + Py\right] = Q e^{\int P dx}$$

$$\frac{d}{dx}\left[y e^{\int P dx}\right] = Q \cdot e^{\int P dx}$$

on Integrating Both sides

$$y e^{\int P dx} = \int Q \cdot e^{\int P dx} + C$$

This is the required solution

Hence the solution is

$$\boxed{y \cdot [I.F] = \int Q \cdot [I.F] \, dx + C}$$

Solve the linear differential equation

$$\frac{dy}{dx} - y = e^{2x}$$

Solution → $P = -1, \quad Q = e^{2x}$

Linear diff eqⁿ $\frac{dy}{dx} + Py = Q$

$I.F = e^{\int p\,dx} = e^{\int -dx} = e^{-x}$

So the general solution is

$$y \cdot [I.F] = \int Q \cdot [I.F]\, dx + C$$

$$y \cdot e^{-x} = \int e^{2x} \cdot e^{-x}\, dx + C$$

$$y = e^{x} \int e^{x}\, dx + c \implies y = e^{x}[e^{x} + c]$$

$$y = ce^{x} + e^{2x}$$

~~Initial~~ Solve $\frac{dy}{dx} + 2y = e^{x}[3\sin 2x + 2\cos 2x]$

<u>Solution</u> here $p = 2, \quad Q = e^{x}[3\sin 2x + 2\cos 2x]$

$I.F = e^{\int p\,dx} = e^{\int 2\,dx} = e^{2x}$

so the general solution is

$$y \cdot [I.F.] = \int Q \cdot [I.F.]\, dx + C$$

$$y \cdot e^{2x} = \int e^{x}[3\sin 2x + 2\cos 2x]\, e^{2x}\, dx + C$$

$$y \cdot e^{2x} = \int e^{3x}[3\sin 2x + 2\cos 2x]\, dx + C$$

$$y \cdot e^{2x} = \int \frac{d}{dx}[e^{3x} \cdot \sin 2x]\, dx + C$$

$$y = e^{-2x} \cdot [e^{3x} \sin 2x + c]$$

$$y = ce^{-2x} + e^{x} \sin 2x \quad \text{Ans.}$$