Example Find the moment generating function of the discrete binomial distribution given by

$P(x) = {}^nC_x\, p^x q^{n-x}$ (where $q = 1-p$)

Also find the first & second moment about the mean

Solution

The moment generating function about the origin is given by $M_x(t) = \sum e^{tx} p(x)$

$$M_x(t) = \sum e^{tx}\, {}^nC_x\, p^x q^{n-x}$$

$$= \sum {}^nC_x (pe^t)^x q^{n-x}$$

$$= (q + pe^t)^n$$

$$\nu_1 = \left[\frac{d}{dt} M_x(t)\right]_{t=0}$$

$$= \left[n(q+pe^t)^{n-1} \cdot pe^t\right]_{t=0}$$

$$= n(p+q)^{n-1} p e^0 \quad \text{Since } q+p=1$$

$$= np$$

$$\nu_2 = \left[\frac{d^2}{dt^2} M_x(t)\right] = \frac{d}{dt}\left[np\, e^t (q+pe^t)^{n-1}\right]$$

$$= \left[np\left\{e^t \cdot (n-1)(q+pe^t)^{n-2} pe^t + (q+pe^t)^{n-1} e^t\right\}\right]$$

$$= \left[np(q+pe^t)^{n-2} \cdot e^t\left\{(n-1)pe^t + (q+pe^t)\right\}\right]_{t=0}$$

$$\nu_2 = \left[np(q+pe^t)^{n-2} \cdot e^t(q+npe^t)\right]_{t=0}$$

$$= np(q+p)\, e^0 (q+np) \qquad [\because q+p=1]$$

$$= np(q+np)$$

$$= npq + n^2p^2$$

Hence first & second moments about the mean are given by

$$\mu_1 = 0$$

Since $\bar{x} = \nu_1 = np$

$$\mu_2 = \nu_2 - \bar{x}^2$$

$$= \nu_2 - \nu_1^2$$

$$= npq + n^2p^2 - n^2p^2$$

$$= npq$$

Hence mean $= np$ & $S.D = \sqrt{\mu_2} = \sqrt{npq}$

Example → Find the moment generating function of discrete Poisson Distribution given by $P(x) = \frac{e^{-\lambda}\lambda^x}{x!}$. Also find the first and seconds moments about mean.

Solution → Moment generating function about the origin is given by

$$M_x(t) = \sum e^{tx} P(x)$$

$$= \sum e^{tx} e^{-\lambda} \frac{\lambda^x}{x!}$$

$$= e^{-\lambda} \sum \frac{(\lambda e^t)^x}{x!}$$

$$\frac{\lambda e^t}{1!} + \frac{(\lambda e^t)^2}{2!} + \frac{(\lambda e^t)^3}{3!} + \cdots = e^{\lambda e^t}$$

$$= e^{-\lambda} \cdot e^{\lambda e^t}$$

$$= e^{\lambda(e^t - 1)}$$

$$\nu_1 = \left[\frac{d}{dt} M_x(t)\right] = \left[e^{\lambda(e^t-1)} \lambda e^t\right]_{t=0}$$

$$= e^{\lambda(1-1)} \lambda e^0$$

$$\nu_2 = \left[\frac{d^2}{dt^2} M_x(t)\right] = \frac{d}{dt}\left[e^{\lambda(e^t-1)} \lambda e^t\right]$$

$$= \left[\lambda\left\{e^t \cdot e^{\lambda(e^t-1)} \cdot \lambda e^t + e^{\lambda(e^t-1)} e^t\right\}\right]_{t=0}$$

$$= \left[\lambda e^{\lambda(e^t-1)} e^t (\lambda e^t + 1)\right]_{t=0}$$

$$= \lambda(\lambda+1)$$

Hence first & second moments about the mean are given by

$$\mu_1 = 0$$

Since $\nu_1 = \bar{x} = \lambda$

$$\therefore \mu_2 = \nu_2 - \bar{x}^2$$

$$= \nu_2 - \nu_1^2$$

$$= \lambda(\lambda+1) - \lambda^2$$

$$= \lambda$$