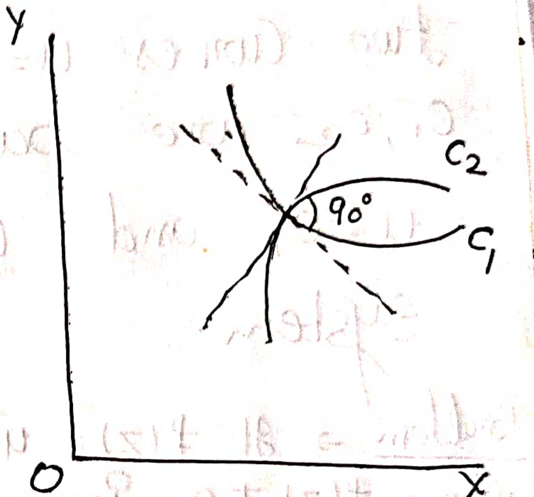
ORTHOGONAL CURVES

→ Two Curves are said to be orthogonal to each other when they intersect at right angle at each of their points of Intersection.

The analytic function $f(z) = u+iv$ Consists of two families of Curves $u(x,y) = C_1$ and $v(x,y) = C_2$ which form an orthogonal system.

$$u(x,y) = C_1 \rightarrow (1)$$

$$v(x,y) = C_2 \rightarrow (2)$$



m_1 = slope of the tangent to the curve $u = C_1$

m_2 = slope of the tangent to the Curve $v = C_2$

If we show that $m_1 m_2 = -1$ the result will be proved.

Taking differentiation of (1) & (2) we get $du=0$, $dv=0$

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} \cdot dy \Rightarrow 0 = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$= \cancel{0 dx + 0 dy = 0}$$

so $\frac{dy}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = m_1$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} \cdot dy = 0$$

$$\frac{dy}{dx} = \frac{-\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}} = m_2$$

$$m_1 m_2 = \left[\frac{-\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}\right]\left[\frac{-\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}}\right]$$

Apply C-R Equations

$$m_1 m_2 = -1$$

Since $m_1 m_2 = -1$

two Curves $u = c_1$ and $v = c_2$ are orthognal and c_1, c_2 are parameters.

$u = c_1$ and $v = c_2$ form an orthogonal system.

Problem → If $f(z) = u + iv$ is an analytic function, regular in D where $f(z) \neq 0$ Prove that the Curves $u =$ Const, $v =$ Const. form two orthogonal families. Verify this in Case of $f(z) = \sin z$

Proof → (i) Proof of above theorem

$\boxed{\frac{\partial}{\partial y}(\sin hy) = \cos hy}$ $\boxed{\frac{\partial}{\partial y}(\cos hy) = \sin hy}$

$\boxed{\begin{array}{l}\cos iy = \cos hy \\ \sin iy = i \sin hy\end{array}}$

II → $f(z) = \sin z$

$u + iv = f(z) = \sin z = \sin(x + iy)$

$= \sin x . \cos iy + \cos x \sin iy$

$= \sin x \cos hy + i \cos x \sin hy$

$u = \sin x \cos hy = c_1$ say

$v = \cos x \sin hy = c_2$ say

Differentiate Both w.r.t x

$\cos x \cos hy + \sin x \sin hy \left(\frac{dy}{dx}\right)_1 = 0$

$-\sin x \sin hy + \cos x \cos hy \left(\frac{dy}{dx}\right)_2 = 0$

$$\left(\frac{dy}{dx}\right)_1 = \frac{\cos x \cos hy}{-\sin x \sin hy}$$

$$\left(\frac{dy}{dx}\right)_2 = \frac{\sin x \sin hy}{\cos x \cos hy}$$

Multiply these two

$$\left(\frac{dy}{dx}\right)_1 \left(\frac{dy}{dx}\right)_2 = -1$$

Proved