...thogonal function $\rightarrow$ Two functions $f_1(x)$ and $f_2(x)$, continuous on the Interval $[a, b]$ are said to be orthogonal on $[a, b]$ if

$$\int_a^b f_1(x)\, f_2(x)\, dx = 0$$

Orthonormal function The set $\{f_i(x)\}$ is said to be orthonormal if

$$(f_i, f_j) = \int_a^b f_i(x)\, f_j(x)\, dx = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

Theorem-2 $\Rightarrow$ The eigen function of a symmetric Kernel Corresponding to different eigen values are orthogonal.

Proof $\Rightarrow$ let $\varphi_m(x)$ and $\varphi_n(x)$ are eigen function of a symmetric Kernel $K(x,t)$ for eigen value λ_m & λ_n respectively of the homogeneous Fredholm integral Equation

$$\varphi(x) = \lambda \int_a^b K(x,t)\, \varphi(t)\, dt \rightarrow (1)$$

Also Kernel $K(x,t)$ is symmetric so $K(x,t) = \overline{K(t,x)}$ $\rightarrow (2)$

Here, we observed that $\lambda = 0$ can not be an eigen value since it gives the trivial solution $\varphi(x) = 0$

Now φ_m & φ_n both satisfy the integral eqn (1)

$$\varphi_m(x) = \lambda_m \int_a^b K(x,t)\,\varphi_m(t)\,dt \rightarrow (3)$$

$$\& \quad \varphi_n(x) = \lambda_n \int_a^b K(x,t)\,\varphi_n(t)\,dt \rightarrow (4)$$

Multiply (3) by $\varphi_n(x)$ & Integrate w.r.t. x over an Interval $[a,b]$

$$\int_a^b \varphi_m(x)\varphi_n(x)\,dx = \lambda_m \int_a^b \varphi_n(x)\left\{\int_a^b K(x,t)\,\varphi_m(t)\,dt\right\}dx$$

Changing order of Integration

$$= \lambda_m \int_a^b \varphi_m(t)\underbrace{\left\{\int_a^b K(x,t)\,\varphi_n(x)\,dx\right\}}_{(5)}dt$$

Since Kernel $K(x,t)$ is symmetric therefore using (2)

(5) can be written as

$$\int_a^b \varphi_m(x)\varphi_n(x)\,dx = \lambda_m \int_a^b \varphi_m(t)\left\{\int_a^b K(t,x)\,\varphi_n(x)\,dx\right\}dt \rightarrow (6)$$

By Interchanging the Variable x & t in (4)

$$\varphi_n(t) = \lambda_n \int_a^b K(t,x)\,\varphi_n(x)\,dx \rightarrow (7) \Rightarrow \frac{\varphi_n(t)}{\lambda_n} = \int_a^b K(t,x)\,\varphi_n(x)\,dx$$

using relation (7), then Eq^n (6) reduces to using

$$\int_a^b \varphi_m(x)\,\varphi_n(x)\,dx = \frac{\lambda_m}{\lambda_n}\int_a^b \varphi_m(t)\,\varphi_n(t)\,dt$$

$$\int_a^b \varphi_m(x)\,\varphi_n(x)\,dx = \frac{\lambda_m}{\lambda_n}\int_a^b \varphi_m(x)\,\varphi_n(x)\,dx$$

$$\left(1-\frac{\lambda_m}{\lambda_n}\right)\int_a^b \varphi_m(x)\,\varphi_n(x)\,dx = 0 \rightarrow (8)$$

Since $\lambda_m \neq \lambda_n$ therefore (8) gives

$$\int_a^b \varphi_m(x)\,\varphi_n(x)\,dx = 0$$

Therefore, We can say that if $\varphi_m(x)$ & $\varphi_n(x)$ are Eigen functions of (1) corresponding to distinct eigen Values, then φ_m & φ_n are orthogonal over the interval $[a,b]$. Hence, the Eigen functions (corresponding to different eigen Values are orthogonal.