Solution of Valterra Integral Equation of the second Kind by successive approximation:

Neumann Series

Theorem →

let $u(x) = f(x) + \lambda \int_a^x K(x,t)\, u(t)\, dt \rightarrow (1)$

be the given Valterra integral Equation of second Kind, such that

(i) $K(x,t)$ is real and Continuous in a rectangle R for which $a \leq x \leq b$, $a \leq t \leq b$

$|K(x,t)| \leq M$ in R $[K(x,t) \neq 0]$

(ii) $f(x) \neq 0$ is real and Continuous in I ; $a \leq x \leq b$

(iii) λ is a non-Zero Constant

The Equation (1) has one and only one Continuous solution $u(x)$ in I which is given by absolutely and uniformly Convergent series

$$u(x) = f(x) + \lambda \int_a^x K(x,t) f(t)\, dt + \lambda^2 \int_a^x K(x,t) \int_a^t K(t,t_1) f(t_1)\, dt_1\, dt + \cdots$$

$$= f(x) + \sum_{m=1}^{\infty} \int_a^x K_m(x,t) f(t)\, dt$$

Proof → Taking any continuous approxim... $u_0(x)$ in (a,b)

Then we have

4. $$u_1(x) = f(x) + \lambda \int_a^x K(x,t)\, u_0(t)\, dt$$

or

$$u_1(x) = f(x) + \lambda \int_a^x K(x,t_1)\, u_0(t_1)\, dt_1$$

$$u_1(t) = f(t) + \lambda \int_a^t K(t,t_1)\, u_0(t_1)\, dt_1$$

If $u_n(x)$ is the nth approximation, then we have

$$u_n(x) = f(x) + \lambda \int_a^x K(x,t)\, u_{n-1}(t)\, dt \rightarrow (2)$$

$$u_2(x) = f(x) + \lambda \int_a^x K(x,t)\, u_1(t)\, dt$$

$$u_2(x) = f(x) + \lambda \int_a^x K(x,t) \left[f(t) + \lambda \int_a^t K(t,t_1)\, u_0(t_1)\, dt_1 \right] dt$$

or

$$u_2(x) = f(x) + \lambda \int_a^x K(x,t_1) \left[f(t_1) + \lambda \int_a^{t_1} K(t_1,t_2)\, u_0(t_2)\, dt_2 \right] dt_1$$

Similarly

$$u_3(x) = f(x) + \lambda \int_a^x K(x,t)\, u_2(t)\, dt$$

$$= f(x) + \lambda \int_a^x K(x,t) \left[f(t) + \lambda \int_a^t K(t,t_1)\, f(t_1)\, dt_1 + \lambda^2 \int_a^t \int_a^{t_1} K(t,t_1)\, K(t_1,t_2)\, u_0(t_2)\, dt_2\, dt_1 \right] dt$$

$$= f(x) + \lambda \int_a^x K(x,t) f(t)\,dt + \lambda^2 \int_a^x \int_a^t K(x,t) K(t,t_1) f(t_1)\,dt_1\,dt$$

$$+ \lambda^3 \int_a^x \int_a^t \int_a^{t_1} K(x,t) K(t,t_1) K(t_1,t_2) u_0(t_2)\,dt_2\,dt_1\,dt$$

Proceeding in the same way, we get

$$u_n(x) = f(x) + \lambda \int_a^x K(x,t) f(t)\,dt + \lambda^2 \int_a^x \int_a^{t_1} K(x,t) K(t,t_1) f(t_1)\,dt_1\,dt + \cdots$$

$$+ \cdots + \lambda^{n-1} \underbrace{\int_a^x \cdots \int_a^{t_{n-3}}}_{(n-1)\text{ times}} K(x,t) K(t,t_1) \cdots K(t_{n-3},t_{n-2})\,dt_{n-2} \cdots dt + R_{n+1} \quad \text{③}$$

where

$$R_{n+1} = \lambda^n \underbrace{\int_a^x \cdots \int_a^{t_{n-2}}}_{n\text{ times}} K(x,t) \cdots K(t_{n-2},t_{n-1})\, u_0(t_{n-1})\,dt_{n-1} \cdots dt \quad \text{④}$$

Now since $|K(x,t)| \leq M$ on $[a,b]$ given in theorem's statement

$$\therefore |R_{n+1}| \leq |\lambda^n| \cdot M^n \underbrace{\int_a^x \cdots \int_a^{t_{n-1}}}_{n\text{ times}} u_0(t_{n-1}) \cdots dt_{n-1} \cdots dt \quad (i)$$

Also, $u_0(x)$ is continuous, therefore it is bounded [$\because$ every continuous function is bounded]

Let $|u_0(x)| \le U$ in (a, b)

Then $|R_{n+1}(x)| \le |\lambda|^n M^n . U \dfrac{(x-a)^n}{n!}$

$$\le |\lambda|^n . M^n U \frac{(b-a)^n}{n!} \quad [\because a \le x \le b]$$

$$|R_{n+1}(x)| \le \frac{[|\lambda| M (b-a)]^n U}{n!}$$

letting $n \to \infty$ we get

$$\lim_{n \to \infty} R_{n+1}(x) = 0$$

Now Consider

$$f(x) + \lambda \int_a^x K(x,t) f(t)\, dt + \cdots + \lambda^{n-1} \underbrace{\int_a^x \cdots \int_a^x}_{(n-1)\text{ times}} K(x,t) \cdots K(t_{n-3}, t_{n-2}) f(t_{n-2})\, dt_{n-2} \cdots dt \quad \text{(5)}$$

The general term of the series (5) is given by

$$S_n = \lambda^{n-1} \underbrace{\int_a^x \cdots \int_a^x}_{(n-1)\text{ times}} K(x,t) \cdots K(t_{n-3}, t_{n-2})\, dt_{n-2} \cdots dt$$

let $|f(t_{n-2})| < N$

Then $|S_n| \le \dfrac{|\lambda| M (b-a)^{n-1}}{(n-1)!} . N$ [as Previous] on Top of this page

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$$v_n = \frac{[|\lambda| M(b-a)]^{n-1}}{(n-1)!}$$

then

$$\frac{v_n}{v_{n+1}} = \frac{[|\lambda| M(b-a)]^{n-1}}{(n-1)!} \cdot \frac{n!}{[|\lambda| M(b-a)]^n}$$

$$\frac{v_n}{v_{n+1}} = \frac{n}{|\lambda| \cdot M(b-a)} \qquad \because n! = n(n-1)!$$

$$\lim_{n\to\infty} \frac{v_n}{v_{n+1}} = \infty$$

$\Rightarrow$ v_n is convergent ~~series~~.

Hence ΣS_n is uniformly convergent

Since each term of this uniformly convergent series is continuous, therefore limit function is also continuous

let $\lim_{n\to\infty} u_n(x) = v(x)$

then $$v(x) = f(x) + \lambda \int_a^x K(x,t) f(t)\, dt \quad \text{(6)}$$

Now, we shall show that $v(x)$ is a solution of (1)

Multiply both sides of (6) by $\lambda K($ and integrate from a to x, we get

$$\lambda \int_a^x K(x,t)\, v(t)\, dt = \lambda \int_a^x K(x,t)\, f(t)\, dt + \lambda^2 \int_a^x \int_a^x K(x,t)\, K(t,t_1)\, f(t_1)\, dt_1\, dt + \cdots$$

Then by (6) we have

$$\lambda \int_a^x K(x,t)\, v(t)\, dt = v(x) - f(x)$$

$$\Rightarrow v(x) = f(x) + \lambda \int_a^x K(x,t)\, v(t)\, dt$$

Hence, $v(x)$ given by (6) is the solution of (1)

Uniqueness $\to$ Now, we shall show that $v(x)$ is unique

Let, if possible $\phi(x)$ be any other solution of (1) then

let $u_0(x) = \phi(x)$

$$u_1(x) = f(x) + \int_a^x K(x,t)\, \phi(t)\, dt = \phi(x) \quad \to (7)$$

i.e $u_1(x) = \phi(x)$

Similarly $u_2(x) = \phi(x)$, $u_3(x) = \phi(x)$ — — and so on which implies that limit function of $u_n(x)$ will be $\phi(x)$.

But since limit is independent of the choice of $u_0(x)$, therefore $\phi(x) = v(x)$. Hence Solution is unique.