Helmholtz Vorticity Equation :- If the external forces are conservative and density is a function of pressure only then

$$\frac{d}{dt}\left(\frac{\vec{w}}{\rho}\right) = \left(\frac{\vec{w}}{\rho}\cdot\nabla\right)\vec{q}$$

Proof :- F is conservative $\vec{F} = -\nabla\Omega$

density ρ is a function of pressure p only

$\Rightarrow$ there exist a relation of the type

$$P = -\int_{p_0}^{p} \frac{dp}{\rho}$$

$$\Rightarrow \nabla P = \sum i \frac{\partial P}{\partial x} = \sum i \frac{dP}{dp}\cdot\frac{\partial p}{\partial x}$$

$$= \sum \frac{1}{\rho} i \frac{\partial p}{\partial x}$$

$$\Rightarrow \nabla P = \frac{1}{\rho}\nabla p \rightarrow (*)$$

By Euler's equation of motion

$$\frac{d\vec{q}}{dt} = F - \frac{1}{\rho}\nabla p$$

using (*)

individuals local rate of change

$$\frac{\partial \vec{q}}{\partial t} + (\vec{q}\cdot\nabla)\vec{q} = -\nabla\Omega - \nabla P$$

But $\nabla(\vec{q}\cdot\vec{q}) = 2\left[\vec{q}\times \text{curl}\,\vec{q} + (\vec{q}\cdot\nabla)\vec{q}\right]$

$$\nabla\left(\frac{\vec{q}^2}{2}\right) - \vec{q}\times \text{curl}\,\vec{q} = (\vec{q}\cdot\nabla)\vec{q}$$

$$\frac{\partial \vec{q}}{\partial t} + \nabla\left(\frac{\vec{q}^2}{2}\right) - \vec{q}\times \operatorname{curl}\vec{q} = -\nabla(\Omega + P)$$

$$\because W = \operatorname{curl}\vec{q}$$

$$\frac{\partial \vec{q}}{\partial t} + \nabla\left[\Omega + P + \frac{1}{2}\vec{q}^2\right] = q \times W$$

Taking curl of both sides & noting that

curl grad of any vector $= 0$

$$\operatorname{curl}\frac{\partial \vec{q}}{\partial t} + \operatorname{curl}\nabla\left[\Omega + P + \frac{1}{2}\vec{q}\right] = \operatorname{curl} \vec{q}\times\vec{W} \qquad \left(\operatorname{curl}\frac{\partial \vec{q}}{\partial t} = \frac{\partial}{\partial t}\operatorname{curl}\vec{q} = \frac{\partial}{\partial t}\vec{W}\right.$$

$$\frac{\partial \vec{W}}{\partial t} = \operatorname{curl}(\vec{q}\times\vec{W})$$

$$\text{or } \frac{\partial \vec{W}}{\partial t} = q\,(\underbrace{\nabla\cdot\vec{W}}_{=0}) - W(\nabla\cdot\vec{q}) + (W\cdot\nabla)\vec{q} - (q\cdot\nabla)\vec{W}$$

But $\nabla\cdot W = \operatorname{div} W = \operatorname{div}\operatorname{curl}\vec{q} = 0$

& Equation of Continuity is $\frac{\partial \rho}{\partial t} + \rho(\nabla\cdot\vec{q}) = 0$

so $\nabla\cdot q = -\frac{1}{\rho}\frac{\partial \rho}{\partial t}$

$$\text{so } \frac{\partial \vec{W}}{\partial t} + \frac{W}{\rho}\frac{\partial \rho}{\partial t} + (W\cdot\nabla)\vec{q} - (q\cdot\nabla)\vec{W}$$

$$\left(\frac{\partial}{\partial t} + q\cdot\nabla\right)\vec{W} = \frac{W}{\rho}\frac{\partial \rho}{\partial t} + (W\cdot\nabla)\vec{q}$$

$$\frac{d\vec{W}}{dt} = \frac{W}{\rho}\frac{\partial \rho}{\partial t} + (W\cdot\nabla)\vec{q}$$

$$\text{or } \frac{1}{\rho}\frac{d\vec{W}}{dt} - \frac{W}{\rho^2}\frac{\partial \rho}{\partial t} = \frac{1}{\rho}(W\cdot\nabla)\vec{q}$$

$$\text{or } \frac{d}{dt}\left(\frac{W}{\rho}\right) = \left(\frac{W}{\rho}\cdot\nabla\right)\vec{q}$$

This is called Helmholtz Vorticity Equation