*) The Normed space $\mathbb{R}^n$ & $\mathbb{C}^n$ are Banach spaces.

<u>Proof</u>:- $(\mathbb{R}^n, \|\cdot\|)$

where $\|x\| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$

We consider any Cauchy sequence (x_m) in $\mathbb{R}^n$. writing

$$x_m = \left(\xi_1^{(m)}, \xi_2^{(m)}, \ldots \xi_n^{(m)}\right).$$

Since (x_m) is Cauchy, then for every $\epsilon > 0$ $\exists$ a natural N

Such that

$$\|x_m - x_n\| = \left(\sum_{i=1}^{n} \left(\xi_i^{(m)} - \xi_i^{(n)}\right)^2\right)^{1/2} < \epsilon \quad \text{for } m, r > N.$$

Squaring both side we get

$$\left(\xi_i^{(m)} - \xi_i^{(n)}\right)^2 < \epsilon^2 \quad \text{for } m, r > N$$
$$\& \; j = 1 \text{ to } n.$$

This shows that for each fixed i $(1 \le i \le n)$, the sequence $\left(\xi_i^{(1)}, \xi_i^{(2)}, \xi_i^{(3)}, \ldots\right)$ is cauchy sequence of real number.

Since $\mathbb{R}$ is a complete space.

so. sequence $\xi_i^{(m)} \longrightarrow \xi_i$ as $m \to \infty$.

Using these limits, we define

$$x = (\xi_1, \xi_2, \xi_3, \dots \xi_n) \in \mathbb{R}^n.$$

Hence. $\|x_m - x\| \leq \epsilon \qquad \forall m \geq N.$

A) Compeletness of l_∞ :

$$\|x\| = \sup_i |x_i|$$

Proof— Let (x_m) be any Cauchy sequence in the space l_∞, where $(x_m) = (\xi_1^{(m)}, \xi_2^{(m)}, \xi_3^{(m)}, \ldots)$.

Since the norm is

$$\|x\| = \sup_i |\xi_i|.$$

where $x = (\xi_i)$. Since the taken sequence (x_m) is Cauchy, for any $\epsilon > 0$ $\exists$ an N such that

$$\|x_m - x_n\| = \sup_i |\xi_i^{(m)} - \xi_i^{(n)}| < \epsilon$$

for all $m, n > N$.

For a fixed i,

$$|\xi_i^{(m)} - \xi_i^{(n)}| < \epsilon \quad \text{——} (*)$$

Hence for every fixed i, the sequence $(\xi_i^{(1)}, \xi_i^{(2)}, \ldots)$ is a Cauchy seq of real numbers.

Since $\mathbb{R}$ is complete, we have.

$$\xi_i^{(m)} \longrightarrow \xi_i \quad \text{as } m \to \infty.$$

Using these infinitely time limits $\xi_1, \xi_2, \xi_3, \dots$, we define sequence. $x = (\xi_1, \xi_2, \xi_3, \dots)$.

To show $x \in l_\infty$ and $x_m \to x$.

then from $(*)$ with $n \to \infty$ we have

$$|\xi_i^{(m)} - \xi_i| \leq \epsilon. \quad m > N.$$

Since $(x_m) = (\xi_i^{(m)}) \in l_\infty$.

there is a real number $K_m \in \mathbb{R}$

Such that

$$|\xi_i^{(m)}| \leq K_m \quad \text{for all } i.$$

Now

$$|\xi_i| \leq |\xi_i - \xi_i^{(m)}| + |\xi_i^{(m)}|$$

$$\leq \epsilon + K_m. \quad (m > N)$$

$(\xi_i) \in l^\infty$.

$$(-1)^n$$

$$d(x, y) = |x - y|$$

$$x = (x_1, x_2, x_3 \ldots)$$

$$y = (y_1, y_2, y_3 \ldots)$$

$$\|x - y\| = (x_1 - y_1, x_2 - y_2, x_3 - y_3 \ldots)$$

$$\left(1 + \frac{1}{n}\right)$$

$$\sum u_n$$

$$\boxed{\lim u_n = 0}$$

#

$x = (\xi_i) \in \bar{C}$

$(x_n) = (\xi_i^{(n)}) \in C$

$\bar{C} = C$

$C \cup C'$

$(x_n) \longrightarrow x$.

$C \subseteq C \cup C'$

$\bar{C}$

$\bar{C} \subseteq C$

$C \overset{!}{=} \bar{C}$.

for all $n \geq N$ & i, we have.

$$\left|\xi_i^{(n)} - \xi_i\right| \leq \|x_n - x\| < \frac{\epsilon}{3}.$$

$x = (\xi_i)$.

Since $x_N \in C$, its terms $\xi_i^{(N)}$ form a convergent sequence. Such a sequence is Cauchy. Hence $\exists$ an N_1 such that

$$\left|\xi_i^{(N)} - \xi_k^{(N)}\right| < \frac{\epsilon}{3}.$$

$i, k \geq N_1$, the following inequality

$$|\xi_i - \xi_k| \leq \left|\xi_i - \xi_i^{(N)}\right| + \left|\xi_k + \xi_k^{(N)}\right| + \left|\xi_i^{(N)} - \xi_k^{(N)}\right| < \epsilon.$$

$C[a,b]$

ℓ^p. $\qquad \|x\| = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p}$ $\qquad \sum_{n=1}^{\infty} \frac{1}{n^p}$ $\quad (u_n) \quad \frac{1}{n^2}$.

$p = 1$ $\qquad \sum u_n < \infty$

$\left\{ (x_i)_{i\in\mathbb{N}} : \sum |x_i|^p < \infty \right\} = \ell^p$ $\qquad \int_1^{\infty} \frac{1}{x^2}\,dx$ $\quad -\frac{1}{x}\Big|_1^{\infty} = 1$ $\quad \boxed{\sum \frac{1}{n}}$

$\int_1^{\infty} f(x) \leftrightarrows \sum_1^{\infty} f(n)$ $\downarrow$ $\qquad \sum u_n \quad \sum v_n$.

$\int_1^{\infty} \frac{1}{x}\,dx = (\ln x)_1^{\infty}$ $\qquad \lim \frac{u_n}{v_n} = \lambda \lessgtr 1$

Proof:-

Let (x_m) be a Cauchy seq. in the space l^p, where $x_m = (\xi_1^{(m)}, \xi_2^{(m)}, \xi_3^{(m)}, \ldots)$. Then for every $\epsilon > 0$ there is an N such that for all $m, n > N$

$$\|x_m - x_n\| = \left(\sum |\xi_i^{(m)} - \xi_i^{(n)}|^p\right)^{1/p} < \epsilon$$

It follows that for every $i = 1, 2, \ldots$ we have

$$|\xi_i^{(m)} - \xi_i^{(n)}| < \epsilon.$$

We choose a fixed i. Then we see that $(\xi_i^{(1)}, \xi_i^{(2)}, \xi_i^{(3)}, \ldots)$ is a Cauchy sequence of numbers. It converges since $\mathbb{R}$ & $\mathbb{C}$ are complete.

So $\xi_i^{(m)} \longrightarrow \xi_i$ as $m \to \infty$.

Using these limits, we define

$$x = (\xi_1, \xi_2, \xi_3, \ldots)$$

and show that $x \in l^p$ and $x_m \to x$.

Now for $m, n > N$

$$\sum_{i=1}^{k} |\xi_i^{(m)} - \xi_i^{(n)}|^p < \epsilon^p$$

Letting $n \to \infty$, we obtain $m > N$

$$\sum_{i=1}^{k} |\xi_i^{(m)} - \xi_i|^p \leq \epsilon^p.$$

We may now let $K \to \infty$, then for $m \geq N$

$$\sum_{i=1}^{\infty} |\xi_i^{(m)} - \xi_i|^p \leq \epsilon^p \quad (*)$$

$$x_m - x = (\xi_i^{(m)} - \xi_i) \in l^p.$$

Since $x_m \in l^p$, it follows that by the mean of Minikowsky inequality, that

$$x = x_m + (x - x_m) \in l^p.$$

$$x_m \to x.$$