Linear Transformation :

$$T : V_1 \longrightarrow V_2 .$$

$$\Rightarrow : T(x_1 + x_2) = T(x_1) + T(x_2)$$

$$\& \; T(\alpha x_1) = \alpha T(x_1)$$

where $\alpha \in \mathbb{F}$

$x_1, x_2 \in V_1$,

$$\|T\| = \sup_{\substack{x \in V_1 \\ x \neq 0}} \frac{\|T(x)\|_2}{\|x\|_1} .$$

$D(T) \subset X$

$R(T) \subset Y$

$$T : (V_1, \|\cdot\|_1) \longrightarrow (V_2, \|\cdot\|_2)$$

$$T : D(T) \longrightarrow \overset{R(T)}{Y}$$

$$\|T\| = \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|T(x)\|_2}{\|x\|_1}$$

Ex:-

1) Identity Operator. $I : X \longrightarrow X \ni : I(x) = x \ \forall x \in X.$

2) Zero operator. $O : X \longrightarrow Y \ni : O(x) = 0 \ \forall x \in X.$

3) Differentiation operator. Let $X \longrightarrow$ V.S. of poly. on $[a,b]$

$T : (X) \longrightarrow X$

$\ni : T(x(t)) = x'(t).$

4) Integration operator.

$$T : C[a,b] \longrightarrow C[a,b].$$

$$\ni : \quad T(x(t)) = \int_a^t x(\tau)\, d\tau$$

$f^{-1} : B \longrightarrow A$ $\longrightarrow$ $f^{-1} : R(f) \to A.$

$\checkmark\ f : A \longrightarrow B$

$\mathrm{Ker} f = \{0\}.$

1-1 $\longrightarrow$ $\checkmark\ f(x_1) = f(x_2)$

$\Rightarrow \boxed{x_1 = x_2}$

$\forall\ y \in B$

$\exists\ x \in A.$

$\ni : \boxed{f(x) = y}$

Onto $\longrightarrow$ $y \in B$ $\quad f(x) = y$

$x = f^{-1}(y).$

$R(f) = B$

$\boxed{f^{-1} : R(f) \longrightarrow A}$

Bounded linear operator $\left.\right\}$ $T : D(T) \to Y$

where $D(T) \subset X$.

then $T \longrightarrow$ Bounded linear operator.

if $\exists$ a $c \in \mathbb{R}$ such that $\forall x \in D(T)$.

$$\boxed{\|T(x)\| \leq c\,\|x\|} \quad \text{—— (1).}$$

$$\|T\| = \sup_{\substack{x \in D(T) \\ x \neq 0}} \frac{\|T(x)\|}{\|x\|}.$$

$D(T) = \{0\}$.

$$\boxed{\|T\| = 0}$$

$$\boxed{\|T(x)\| \le \|T\|\,\|x\|.}$$

$\|\alpha x\| = |\alpha|\,\|x\|$

<u>Lemma</u>:- Let T be a bounded l.O. as defined above, Then

a) An alternative formula for the norm of T is

$$\boxed{\|T\| = \sup_{\substack{x \in D(T) \\ x = 1}} \|T(x)\|.}$$

$$\|T\| = \sup_{\substack{x \in D(T) \\ x \neq 0}} \frac{\|T(x)\|}{\|x\|}$$

$\|\cdot\| : X \to \mathbb{R}$

b) The <u>norm defind on T</u>. satisfies all conditions of norm.

Proof :- We write $\|x\| = a$ & set $y = (1/a)x$. where $x \neq 0$. Then

$$\|y\| = \left\|\frac{1}{a}x\right\| = \left(\frac{1}{a}\right)\|x\| = \cancel{\frac{1}{a} \times a} = 1.$$

and since T is linear, we have.

$$\|T\| = \sup_{\substack{x \in D(T) \\ x \neq 0}} \frac{\|T(x)\|}{\|x\|} = \sup_{\substack{x \in D(T) \\ x \neq 0}} \left\| T\left(\frac{1}{a}x\right)\right\|.$$

$$= \sup_{\substack{y \in D(T) \\ \|y\| = 1}} \|T(y)\|.$$

① $\|v\| \geq 0$.

② $\|v\| = 0$ iff $v = \theta$.

1) $\|T\| \geq 0$.

$$\|T\| = \sup_{\substack{x \in D(T) \\ \|x\| = 1}} \|T(x)\|$$

≥ 0.

$\|T\| = 0$

$\sup \|T(x)\| = 0$

$\|T(x)\| = 0$

$T(x) = 0$.

$\boxed{T = 0}$

$T = 0$.

then $T(x) = 0 \quad \forall x$.

$\|T(x)\| = 0, \forall x$

$\boxed{\sup \|T(x)\| = 0.}$

$\|T\| = 0$

$$\|\alpha v\| = |\alpha| \, \|v\|.$$

$$\|v_1 + v_2\| \le \|v_1\| + \|v_2\|$$

$$\to \underline{\|\alpha T\|} = |\alpha| \, \|T\|.$$

$$\|\alpha T\| = \sup_{\substack{x \in D(T) \\ \|x\| = 1}} \|(\alpha T(x))\| = \sup |\alpha| \, \|T(x)\|.$$

$$= |\alpha| \sup \|T(x)\|$$

$$= |\alpha| \, \|T\|$$

$x \to a \quad \underline{f(x) \to f(a).}$

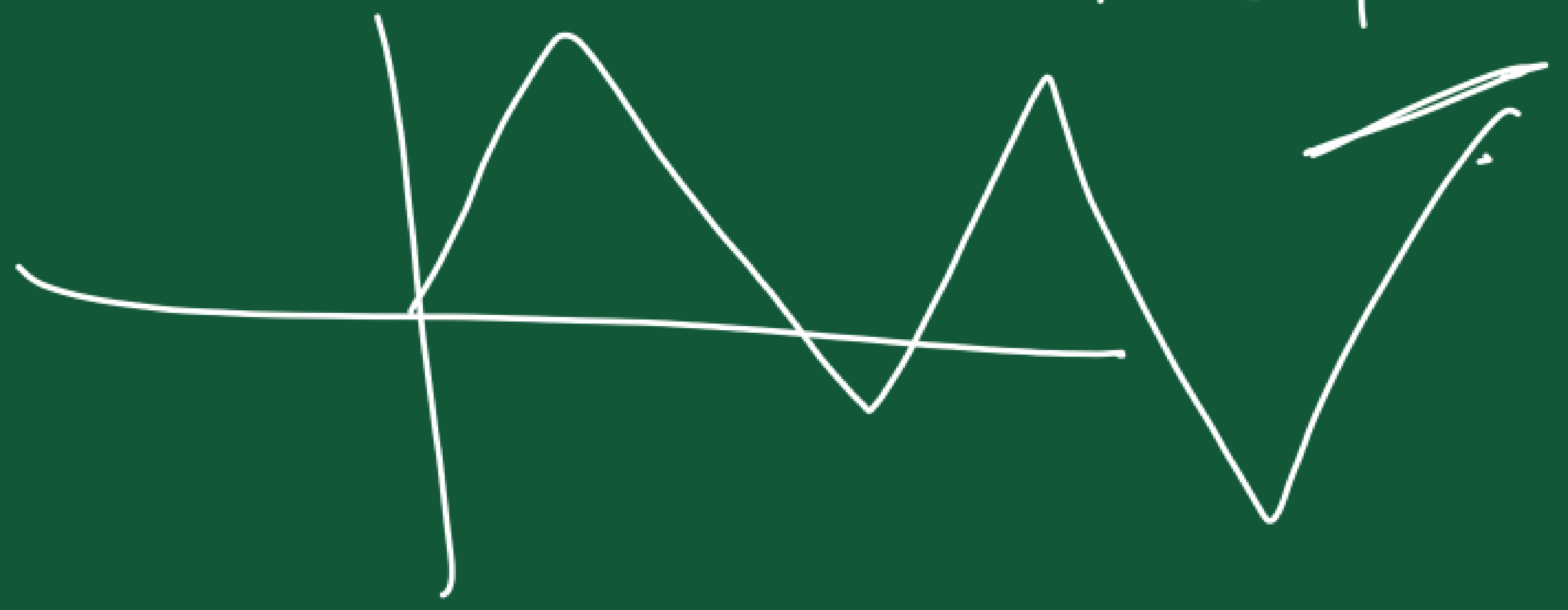
4 $\quad \|T_1 + T_2\| \leq \|T_1\| + \|T_2\|.$

$$\sup_{\substack{x \in D(T_1+T_2) \\ x = 1}} \frac{\|T_1 + T_2(x)\|}{\|x\|} \qquad f(x) = \|x\|$$

$$= \sup_{\substack{x \in D(T_1+T_2) \\ x = 1}} \frac{\|T_1(x) + T_2(x)\|}{\|x\|}.$$

$$\leq \sup_{\substack{x \in D(T_1) \\ \|x\| = 1}} \frac{\|T_1(x)\|}{\|x\|} + \sup \frac{\|T_2(x)\|}{\|x\|}$$

$$= \|T_1\| + \|T_2\|$$



if $x \longrightarrow a$.

$$\boxed{\lim_{x \to a} f(x) = f(a)}$$

$\underline{\epsilon \cdot \delta} \longrightarrow$

whenever

$|f(x) - f(y)| < \epsilon$ $|x - y| < \delta$.

f at 'a'

if $\forall \epsilon > 0 \ \exists \ \underline{\delta}$.

$\Rightarrow$

$[\quad]$

$\delta = f(a, \epsilon)$