Continuous Linear Transformation —

Let N and N' $\longrightarrow$ Normed linear spaces.

$T : N \longrightarrow N'$ $\longrightarrow$ linear transforⁿ.

i.e. if any sequence $x_n \longrightarrow x$ in N

$\Rightarrow$ $T(x_n) \longrightarrow T(x)$ in N'.

Theorem:- Let N and N' be normed li. sp. and T a l.T. from N to N'. Then the following conditions on T are equivalent.

$$\|T\| = \sup_{x \in X} \frac{\|T(x)\|}{\|x\|}.$$

1) T is continuous.

2) T is continuous at the origin, in the sense that $x_n \to 0 \Rightarrow T(x_n) \to 0$.

3) $\exists$ a real number $K \geq 0$ with the property that $\|T(x)\| \leq K\|x\|$ $\forall x \in N$.

4) if $S = \{x : \|x\| < 1\}$ is the closed unit sphere in N

Then its image $\underline{T(S)}$ is a bounded set in N'.

Proof:-

$(1) \Leftrightarrow (2)$. If T is continuous. Then since $T(0)=0$ it is certainly continuous at the origin. On the other hand, if T is continuous at the origin then

$x_n \to x \Leftrightarrow x_n - x \to 0 \Rightarrow T(x_n - x) \to 0 . \Leftrightarrow T(x_n) - T(x) = 0$

$\Leftrightarrow T(x_n) \to T(x)$

$(2) \Leftrightarrow (3)$.

It is obvious that $(3) \Rightarrow (2)$, for if such a K exists then $x_n \to 0$, clearly implies that $T(x_n) \to 0$.

To show that $(2) \Rightarrow (3)$ we assume that $\nexists$ a $K \geq 0$

Then it follows from this that for each positive integer n we can find a vector x_n such that $\|T(x_n)\| > n\|x_n\|$ or equivalently, such that $\|T(x_n/n\|x_n\|)\| > 1$. If we now put

$$y_n = \frac{x_n}{n\|x_n\|}$$

Then it is obvious that $y_n \to 0$ but $T(y_n) \not\to 0$ so T is not continuous at the origin. Which is a contradiction.

Hence $\exists\, K > 0$ such that $\|T(x)\| \leq K\|x\|, \;\forall\, x \in N$

$(3) \Leftrightarrow (4)$:

Since a non empty subset of a N.L.S is bounded $\Leftrightarrow$ it is contained in a closed sphere centred at origin. It is evident that $(3) \Rightarrow (4)$; for $\|x\| \leq 1$ Then $\|T(x)\| \leq K$. To show that $(4) \Rightarrow (3)$ we assume that $T(S)$ is contained in a closed sphere of radius K centred at the origin.

If $x=0$ then $T(x)=0$ & clearly $\|T(x)\| \leq K\|x\|$ and if $x \neq 0$ then $\frac{x}{\|x\|} \in S$ and therefore

$$\left\| T\left(\frac{x}{\|x\|}\right) \right\| \leq K.$$

Hence we have $\boxed{\|T(x)\| \leq K\|x\|.}$

Such that
$$\|f\| = \|f_0\|.$$

Hahn-Banach Theorem :

Lemma. Let M be a linear subspace of a normed linear space N and let f be a functional defined on M. If x_0 is a vector not in M, and if

$$M_0 = M + [x_0]$$

is a linear subspace spanned by M & x_0 then f can be extended to a functional f_0 defined on M_0

Proof:-

$$f_0(x+ax_0) = fx + ar_0.$$

$\|f_0\| = 1$

$$-\|x+ax_0\| \le f(x) + ar_0 \le \|x+ax_0\|.$$

$$-f(x) - \|x+ax_0\| \le ar_0 \le -f(x) + \|x+ax_0\|.$$

$$-f\left(\frac{x}{a}\right) - \left\|\frac{x}{a} + x_0\right\| \le r_0 \le -f\left(\frac{x}{a}\right) + \left\|\frac{x}{a} + x_0\right\| \quad (**)$$

$r_0 = f_0(x_0)$

$$|f_0(x+ax_0)| \le \|x+ax_0\|.$$

$\boxed{a \neq 0}$.

$f(x_2) - f(x_1)$

$= f(x_2 - x_1) \le |f(x_2 - x_1)| \le \|f\| \, \|x_2 - x_1\|$

$= \|x_2 - x_1\| = \|(x_2 + x_0) - (x_1 + x_0)\|$

$\le \|x_2 + x_0\| + \|x_1 + x_0\|$

So

$-f(x_1) - \|x_1 + x_0\| \le -f(x_2) + \|x_2 + x_0\|$ —— $(*)$

$\alpha = \sup\{-f(x) - \|x + x_0\|, x \in M\}$.

$\beta = \inf\{-f(x) + \|x + x_0\|, x \in M\}$.

$\alpha \le \beta$

$\alpha \le r_0 \le \beta$.

$$f(x) = g(x) + i\,h(x).$$

$$h(x) = -g(ix).$$

$$\|g\| \leq 1.$$

$$\boxed{f(x) = g(x) - i\,g(ix).}$$

$$\|g_0\| = \|g\|.$$

The eqn

$$f(ix) = i f(x).$$

$$f(ix) = g(ix) + i\,h(ix) \quad \& \quad i f(x) = i(g(x) + i\,h(x))$$
$$= i g(x) - h(x).$$

$$f_0(x) = g_0(x) - i\, g_0(ix).$$

$$f_0(x+y) = f_0(x) + f_0(y)$$

$$f_0(ax) = a f_0(x).$$

$$\|f_0\| = 1.$$

$$|f_0(x)| \le 1.$$

$$f_0(x) = g_0(x). \quad \|g_0\| \le 1$$

$$f_0(ix) = g_0(ix) - i\, g_0(i^2 x)$$
$$= g_0(ix) + i\, g_0(x)$$
$$= i\left(g_0(x) - i\, g_0(ix)\right) = i f_0(x).$$

$$f_0(x) = r e^{i\theta} \quad r > 0.$$

$$|f_0(x)| = r = e^{-i\theta} f_0(x) = f_0(e^{-i\theta} x)$$

$$\|e^{-i\theta} x\| = \|x\| = 1.$$

$$\boxed{f(x) = x_0}$$

$$\boxed{f_0(x) = x_0}$$