Linear Equations with constant coefficients —

We shall now solve higher order linear equations of the type

$$\left(D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \cdots + a_n D'^n\right) z = 0 \qquad \text{——— (1)}$$

where $D = \frac{\partial}{\partial x}$ & $D' = \frac{\partial}{\partial y}$ & a_i's are constants $(1 \le i \le n)$.

Assume that

$$z = \phi(y + mx)$$

where ϕ is an arbitrary f^n and m is a constant

then

$$Dz = m\phi'(y+mx)$$

$$D^2 z = m^2 \phi''(y+mx)$$

and

$$D'z = \phi'(y+mx)$$
$$D'^2 z = \phi''(y+mx)$$

Therefore putting $z = \phi(mx+y)$ in (1) we get

$$(m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots + a_n)\phi^{(n)}(y+mx) = 0$$

Hence if m is the root of the eqn

$$m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots + a_n = 0 \quad \text{—(3)}$$

Hence if $m_1, m_2, \dots m_n$ are the roots of (3), then

$$z = \phi_1(y+m_1x),\ z = \phi_2(y+m_2x)$$
$$\dots\ z = \phi_n(y+m_nx)$$

are the solns of eqn (1), where ϕ_i's $(1 \le i \le n)$ are n arbitrary fn.

G.S. $\Rightarrow$ $z = \phi_1(y+m_1x) + \phi_2(y+m_2x) + \dots + \phi_n(y+m_nx)$

Que: $\dfrac{\partial^2 z}{\partial x^2} + 3\dfrac{\partial^2 z}{\partial x \partial y} + 2\dfrac{\partial^2 z}{\partial y^2} = 0.$

Solⁿ -

$D = \dfrac{\partial}{\partial x}$ $\quad D^2 = \dfrac{\partial^2}{\partial x^2}$ $\quad DD' = \dfrac{\partial}{\partial x} \cdot \dfrac{\partial}{\partial y} = \dfrac{\partial^2}{\partial x \partial y}.$

$D' = \dfrac{\partial}{\partial y}$ $\quad D'^2 = \dfrac{\partial^2}{\partial y^2}$

$z = \phi(y + mx)$

$z = \phi_1(y + (-1)x)$

$= \phi_1(y - x)$

$z = \phi_2(y - 2x)$

$(D^2 + 3DD' + 2D'^2) z = 0$

the auxiliary eqⁿ will be

$m^2 + 3m + 2 = 0.$

Ques: Solve

$$(D^3 - 4D^2D' + 4DD'^2)z = 0$$

$$z = \phi_1(y+0x) + \phi_2(y+2x) + x\phi_3(y+2x)$$

Case of equal roots:

When the auxiliary eqn

$$m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots + a_n = 0$$

has two equal roots then to obtain such a soln consider the eqn

$$(D - mD')^2 z = 0 \quad \text{--- (1)}$$

arising from the factors which gives equal roots.

put

$(D - mD')z = u$, then (1)

becomes

$$(D - mD')u = 0.$$

This has the solⁿ $u = \phi(y + mx)$.

Therefore

$$(D - mD')z = \phi(y + mx)$$

$$\vdots$$

or

$$p - mq = \phi(y + mx)$$

Hence the char. eqⁿ for this are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{\phi(y+mx)}$$

The first two members give

$$m\,dx + dy = 0$$

Integrating,

$$y + mx = a.$$

Hence

$$\frac{dx}{1} = \frac{dz}{\phi(a)}$$

so that

$$x\,\phi(a) + b = z$$

Since b is arb. constant, take $b = \phi_1(a)$

Hence the soln is

$$z = x\,\phi(a) + \phi_1(a)$$
$$= x\,\phi(y+mx) + \phi_1(y+mx).$$

$$z = y^2$$

Soln -

$$(D^3D' - 4D^2D'^2 + 4DD'^3)z = 0$$

soln -

$$DD'(D-2D')^2 z = 0$$

The soln of

$$Dz = 0 \quad \& \quad D'z = 0$$

$$\frac{\partial z}{\partial x} = 0 \qquad \frac{\partial z}{\partial y} = 0$$

$$z = \phi_1(y) \qquad z = \phi_2(x).$$