Partial order set —

M with relation '$\leq$'

1) $a \leq a, \ \forall a \in M$

2) If $a \leq b$ & $b \leq a$

then $a = b$.

3) If $a \leq b$ & $b \leq c$

then $a \leq c$.

$\forall a, b, c \in M$

$m \in M$

$m \leq x \Rightarrow m = x$

$a, b \in X$.

P.O.S

$a \leq b$

$b \leq a$

$x \leq u, \ \forall x \in w$.

$\subseteq$.

Zorn's Lemma :-

Let $M \neq \phi$ be a P.O. set. Suppose that every chain $C \subset M$ has an upper bound. Then M has atleast one maximal element.

Hamel Basis

Schader Basis

Ques- Every vector space $X \neq \{0\}$ has a Hamel basis.

Proof- - Let M be set of all l.g. subsets of X. Since $X \neq \{0\}$, it has an element $x \neq 0$ and $\{x\} \in M$ so that $M \neq \phi$. Set inclusion defines a partial ordering on M. Every chain $C \subset M$ has an upper bound, namely union of all subsets of X which are elements of C.

Now by Zorn's Lemma, M has a maximal element B. We show that B is Hamel Basis of X. Let $Y = \text{span } B$ then Y is a subspace of X and $Y = X$.

Since otherwise $B \cup \{z\}$, $z \in X$, $z \notin Y$ would be a linearly independent set containing B as a proper subset, contrary to the maximality of B.

$f : X \longrightarrow \mathbb{R}$ or $\mathbb{C}$

↳ linear

- $f(x+y) \leq f(x) + f(y)$
- $f(\alpha x) = \alpha f(x)$

sub

Hahn Banach Theorem —

Let X be a real vector space and p a sublinear functional on X. Furthermore, let f be a linear functional which is defined on a subspace Z of X and satisfies

(2) — $f(x) \leq p(x) \quad \forall\, x \in Z$.

Then f has a linear extension $\bar{f}$ from Z to X satisfying

(3) —— $\bar{f}(x) \le p(x), \quad \forall x \in X.$

that is $\bar{f}$ is linear functionl on X, satisfying (3) on X and

$\bar{f}(x) = f(x), \quad \forall x \in Z.$

Proof:- We shall prove:-

a) The set E of all linear extentions g of f satisfying $g(x) \le p(x)$ on their domain $D(g)$ can be partially ordered and Zorn's lemma yeilds a maximal element $\bar{f}$ of E.

b) $\bar{f}$ is defined on the entire space X.

c) An auxiliary relation which was used in (b).

We start with part

a) Let E be the set of all linear extensions g of f satisfying $g(x) \le p(x)$

Clearly, $E \neq \phi$ since $f \in E$.

On E, we can define a partial ordering by $g \le h$ meaning h is an extension of g. that is By definition $D(h) \supset D(g)$ and $h(x) = g(x)$ $\forall x \in D(g)$.

For any chain $C \subset E$, we now define $\hat{g}$ by

$$\hat{g}(x) = g(x) \quad \text{if } x \in D(g)$$

$\hat{g}$ is linear functional, the domain being

$$D(\hat{g}) = \bigcup_{g \in C} D(g).$$

Which is a vector space since C is a chain.

The definition of $\hat{g}$ is unambiguous. Indeed, for an $x \in D(g_1) \cap D(g_2)$ with g_1 & $g_2 \in C$, we have

$$g_1(x) = g_2(x)$$

Since C is a chain. So that $g_1 \le g_2$ or $g_2 \le g_1$. Clearly $g \le \hat{g}$, $\forall g \in C$.

Hence $\hat{g}$ is an upper bound of C.

Since $C \subset E$ was arbitrary, Zorn's Lemma thus implies that E has a maximal element $\bar{f}$. By the def^n of E, this is a linear extension of f which satisfies

$$(*) \quad \bar{f}(x) \leq p(x), \quad \forall x \in D(\bar{f}).$$