b) We now show that $D(\bar{f})$ is all of X.
Suppose that this is not true. Then we call/choose a $y_1 \in X$ such that $y_1 \notin D(\bar{f})$
& consider the subspace Y_1 of X spanned by $D(\bar{f})$ & y_1. Note that $y_1 \neq 0$ since $0 \in D(\bar{f})$. Any $x \in Y_1$ can be written

$$x = y + \alpha y_1$$

This representation is unique. In fact

$$y + \alpha y_1 = \tilde{y} + \beta y_1 \text{ with } \tilde{y} \in D(\bar{f}) \Rightarrow$$

$$y - \bar{y} = (\beta - \alpha) y_1$$

where $y - \bar{y} \in D(\bar{f})$ whereas $y_1 \notin D(\bar{f})$ so that the only sol^n is $y - \bar{y} = 0$ & $\beta - \alpha = 0$

This means uniqueness.

Now a functional g_1 on Y_1 is defined by

$$g_1(y + \alpha y_1) = \bar{f}(y) + \alpha C \qquad (**)$$

where C is a constant. It is not difficult to see that g_1 is linear.
Furthermore, for $\alpha = 0$ we have

$$g_1(y) = \bar{f}(y).$$

Hence g_1 is a proper extension of $\bar{f}$.

that is an extension such that $D(\bar{f})$ is a proper subset of $D(g_1)$.

Consequently, if we can prove that $g_1 \in E$ by showing that

(*) $\underline{\quad\quad}$ $g_1(x) \le p(x), \ \forall x \in D(g_1)$.

This will contradict the maximality of $\bar{f}$ so that $D(\bar{f}) \ne X$ is false and hence $D(\bar{f}) = X$.

(C) Accordingly, we must finally show that g_1 with a suitable c in eqn (**) satisfies eqn (*). We consider any y and z in $D(\bar{f})$. From (*) and

$$p(x+y) \le p(x) + p(y) \quad \forall x, y \in X.$$

We obtain

$$\begin{aligned}\bar{f}(y) - \bar{f}(z) &= \bar{f}(y-z) \\ &\le p(y-z) \\ &= p(y+y_1-y_1-z) \\ &\le p(y+y_1) + p(-y_1-z)\end{aligned}$$

Hence

$$-p(-y_1-z)-\bar{f}(z) \le p(y+y_1)-\bar{f}(y) \quad (***)$$

where y_1 is fixed. Since y does not appear on the left and z not on the right, the inequality continues to hold if we take the sup over $z \in D(\bar{f})$ on the left (say it as m_0) and the infimum over $y \in D(\bar{f})$ on the right (say it as m_1) then $m_0 \le m_1$ and for a c with

$$m_0 \le c \le m_1$$

We have from $(***)$

a) $$-p(-y_1-z)-\bar{f}(z) \le c \quad \forall z \in D(\bar{f})$$

b) $$c \le p(y+y_1)-\bar{f}(y).$$

We prove $(*')$ first for $-$ve α in $(**)$ and then $+$ve α. For $\alpha < 0$ we use (a) with z replaced by $\alpha^{-1}y$ that is,

$$-p\left(-y_1-\frac{1}{\alpha}y\right)-\bar{f}\left(\frac{1}{\alpha}y\right) \le c$$

Multiplication by $-\alpha > 0$ gives

$$\alpha p\left(-y_1 - \frac{1}{\alpha} y\right) + \bar{f}(y) \leq -\alpha C$$

From this and (**), using $y + \alpha y_1 = x$ we obtain the desired inequality

$$\begin{aligned} g_1(x) &= \tilde{f}(y) + \alpha C \\ &\leq \alpha p\left(-y_1 - \frac{1}{\alpha} y\right) \\ &= p(\alpha y_1 + y) \\ &= p(x) \end{aligned}$$

For $\alpha = 0$ we have $x \in D(\tilde{f})$ and nothing to prove. For $\alpha > 0$ we use (b) with y replaced by $\alpha^{-1} y$ to get

$$C \leq p\left(\frac{1}{\alpha} y + y_1\right) - \bar{f}\left(\frac{1}{\alpha} y\right)$$

Multiplication by $\alpha > 0$ gives

$$\alpha C \leq \alpha p\left(\frac{1}{\alpha} y + y_1\right) - \bar{f}(y)$$

$$= px - \bar{f}(y)$$

From this and (**), $g_1(x) = \bar{f}(y) + \alpha C \leq p(x)$ hence the part (c) is proved.

Hahn-Banach Theorem (Generalized form) —

Let X be a real or complexed vector space and p a real valued functional on X which is sublinear. Furthurmore let f be a linear functional which is defined on a subspace Z of X and satisfies

(*) $|f(x)| \leq p(x)$, $\forall x \in Z$.

Then f has a linear extention $\tilde{f}$ from Z to X

Satisfying

(*') $|\tilde{f}(x)| \leq p(x)$. $\forall x \in X$.

Hahn-Banach Theorem (Normed Space) —

Let f be a bounded linear functional on a subspace Z of a normed space X. Then $\exists$ a bounded linear functional $\tilde{f}$ on X which is extension of f to X and has the same norm

$$\|\tilde{f}\|_X = \|f\|_Z .$$

where $\|\tilde{f}\|_X = \sup_{\substack{x \in X \\ \|x\|=1}} |\tilde{f}(x)|$, $\|f\|_Z = \sup_{\substack{x \in Z \\ \|x\|=1}} |f(x)|$.

Proof If $Z = \{0\}$ then $f = 0$ and the extension is $\tilde{f} = 0$. Let $Z \neq \{0\}$. We want to use H.B.T (Generalized form). Hence we must first discover a suitable p. For all $x \in Z$ we have

$$|f(x)| \leq \|f\|_Z \|x\| .$$

This is of the form $(*)$, where

$$p(x) = \|f\|_Z \|x\| .$$

We see that p is defined on all of X.
Furthermore p satisfies the first condition of sublinear since by the triangular inequality

$$p(x+y) = \|f\|_Z \|x+y\| \leq \|f\|_Z (\|x\|+\|y\|)$$
$$= p(x) + p(y).$$

p also satisfies the second condition of sublinearity on X because
$$p(\alpha x) = \|f\|_Z \|\alpha x\| = |\alpha| \|f\|_Z \|x\| = |\alpha| p(x).$$

Hence we can apply the generalized form of H.B.T. and conclude that $\exists$ a linear functional $\tilde{f}$ on X which is an extension of f and satisfies
$$|\tilde{f}(x)| \leq p(x) = \|f\|_Z \|x\|, \quad x \in X.$$

Taking the sup over all $x \in X$ we obtain
$$\|\tilde{f}\|_X = \sup_{\substack{x\in X \\ \|x\|=1}} |\tilde{f}(x)| \leq \|f\|_Z$$

Since under an extension the norm cannot decrease, we also have

$$\|\tilde{f}\|_X = \|f\|_Z \quad \text{as} \quad \|\tilde{f}\|_X \geq \|f\|_Z.$$

Hence the theorem is proved.

<u>Theorem</u> - Let X be a normed space and let $x_0 \neq 0$ be any element of X. Then $\exists$ a bounded linear functional $\tilde{f}$ on X such that

$$\|\tilde{f}\| = 1, \quad \tilde{f}(x_0) = \|x_0\|.$$

<u>Proof</u> - We consider the subspace Z of X consisting of all elements $x = \alpha x_0$ where α is scalar. On Z, we define a linear functional f by

$$\text{(A)} \quad f(x) = f(\alpha x_0) = \alpha \|x_0\|$$

f is bounded and has norm $\|f\| = 1$ because

$$|f(x)| = |f(\alpha x_0)| = |\alpha| \|x_0\| = \|x\|.$$

Hence by previous Theorem $\Rightarrow$ f has a linear extension $\tilde{f}$ from Z to X of

$\|\tilde{f}\| = \|f\| = 1$. From (A) we see that

$$\tilde{f}(x_0) = f(x_0) = \|x_0\|.$$

Theorem – (Corollary) —.

For any x in a normed space X we have

$$\|x\| = \sup_{\substack{f \in X' \\ f \neq 0}} \frac{|f(x)|}{\|f\|},$$

(X^*)

X' = Set of all bounded linear functional defined on X.

Proof – From Previous theorem, we have writing x for x_0,

$$\sup_{\substack{f \in X' \\ f \neq 0}} \frac{|f(x)|}{\|f\|} \geq \frac{|\tilde{f}(x)|}{\|\tilde{f}\|} = \frac{\|x\|}{1} = \|x\| \quad \text{---}(1)$$

and from $|f(x)| \leq \|f\| \, \|x\|$ we obtain

$$\sup_{\substack{f \in X' \\ f \neq 0}} \frac{|f(x)|}{\|f\|} \leq \|x\| \quad \text{---}(2)$$

from (1) & (2) we get the desired result.