Natural Embedding from $X \to X^{**}$

$C : X \to X^{**}$ — onto

$X \to$ V.S.

$$\boxed{C : X \to X^{*}}$$

Isomorphism → linear, bijective, norm preserving

X^*

$f' : X^* \to \mathbb{C}$ or $\mathbb{R}$

$(X^*)^*$

$f : X \to \mathbb{C}$ or $\mathbb{R}$

X^* → vector space

We can obtain a $g \in X^{**}$, which is a linear functional defined on X^*, be choosing a fixed $x \in X$ and setting

$$g(f) = g_x(f) = f(x).$$

(x is fixed, $f \in X^*$ (variable))

$g : X^* \longrightarrow \mathbb{C}$ or $\mathbb{R}$.

$f \in X^*$

$f : X \longrightarrow \mathbb{R}$ or $\mathbb{C}$.

Proposition - g is linear.

$g : X^* \to \mathbb{R}$ or $\mathbb{C}$

$\Rightarrow$: $g(f) = g_x(f) = f(x)$

(Here x is fixed $f \in X^*$)

$$g(\alpha f + \beta f') = \alpha\, g(f) + \beta\, g(f').$$

L.H.S.

$$g(\alpha f + \beta f') = g_x(\alpha f + \beta f') = (\alpha f + \beta f')(x) = \alpha\, f(x) + \beta f'(x)$$

$$= ((\alpha f(x)) + (\beta f'(x)) = \alpha\, g_x(f) + \beta\, g_x(f') = \alpha\, g(f) + \beta\, g(f').$$

To each $x \in X$, there corresponds a $g_x \in X^{**}$. This defines a mapping

$$C : X \longrightarrow X^{**}$$
$$x \longmapsto g_x$$

C is called the cannonical mapping of X into X^{**}.

C is linear since its domain is a v.s. and we have

$$(C(\alpha x + \beta y))(f) = g_{(\alpha x + \beta y)}(f).$$
$$= f(\alpha x + \beta y)$$
$$= \alpha f(x) + \beta f(y)$$
$$= \alpha g_x(f) + \beta g_y(f).$$
$$= \alpha C(x)(f) + \beta C(y)(f).$$
$$= (\alpha C(x) + \beta C(y))(f)$$

$$\Rightarrow \boxed{C(\alpha x + \beta y) = \alpha C(x) + \beta C(y)}$$

Dual Space (X') ——

$X \longrightarrow$ Normed Space

the set of all bounded linear functionals on X $\Big\}$ forms a Normed Space with norm defined by

$$\|f\| = \sup_{\substack{x \in X \\ x \neq 0}} \frac{|f(x)|}{\|x\|} = \sup_{\substack{x \in X \\ \|x\|=1}} |f(x)|$$

Ex-(1) The dual space of $\mathbb{R}^n$ is $\mathbb{R}^n$.

proof:- Since $\mathbb{R}^{n'} = \mathbb{R}^{n*}$ and every $f \in \mathbb{R}^{n*}$ has a representation

$$f(x) = \sum \xi_k \gamma_k .$$

Here $\gamma_k = f(e_k)$

(sum 1 to n)

By the Cauchy-Schwarz inc.

$$|f(x)| \leq \sum |\xi_k \gamma_k|$$

$$\leq \left(\sum \xi_k^2\right)^{1/2} \left(\sum \gamma_k^2\right)^{1/2}$$

$$= \|x\| \left(\sum \gamma_k^2\right)^{1/2}$$

Taking the supremum over all x of norm 1 we obtain

$$\|f\| \leq \left(\sum \gamma_k^2\right)^{1/2}$$

However, since for $x = (\gamma_1, \gamma_2, -- \gamma_k)$ equality is acheived in the C.S.I., we must infact have

$$\|f\| = \left(\sum_{i=1}^{n} \gamma_k^2\right)^{1/2}$$

This proves that the norm of f is the Euclidean norm, and $\|f\| = \|c\|$

where $c = (\gamma_k) \in \mathbb{R}^n$.

Hence the mapping of $\mathbb{R}^{n'}$ onto $\mathbb{R}^n$ defined by $f \mapsto c = (\gamma_k)$, $\gamma_k = f(e_k)$, is norm preserving and since it is linear and bijective, it is an isomorphism.

Ex-2 - The dual space of l^1 is l_∞.

Ex-3 - The dual space of l^p is l_q,

Here $1 < p < \infty$

and q is conjugate of p that is

$$\frac{1}{p} + \frac{1}{q} = 1.$$