Lemma (Cannonical Mapping) (B).

The Cannonical mapping C given by $C : X \longrightarrow X''$ such that $x \longmapsto g_x$ is an isomorphism of the normed space X onto the normed space $R(C)$, the range of C.

Proof. Linearity of C is obvious. Since

$$g_{\alpha x+\beta y}(f) = f(\alpha x+\beta y)$$
$$= \alpha f(x) + \beta f(y)$$
$$= \alpha g_x(f) + \beta g_y(f)$$

In particular,

$$g_x - g_y = g_{x-y}$$. Hence by $\|g_x\| = \|x\|$

we obtain,

$$\| g_x - g_y \| = \| g_{x-y} \| = \| x-y \|.$$

This shows that C is isometric; it preserves the norm. Isometry $\Rightarrow$ injective (one-one).

We can also see if $x \neq y$ then $g_x \neq g_y$. Hence C is one-one & onto regarded as a mapping onto its range.

Definition (Reflexivity) :-

A normed space X is said to be reflexive if

$$R(C) = X''$$

where $C : X \longrightarrow X''$ is the canonical mapping such that $x \longmapsto g_x$ & $g_x(f) = f(x)$.

$C : X \longrightarrow X''$

Theorem - If a normed space X is reflexive, it is complete. (Banach space).

Proof. - Since X'' is the dual space of X', it is complete by result (*). Reflexivity of X means that $R(C) = X''$. Completeness of X now follows from that of X'' by lemma (B).

Result (A) :-

The dual space X' of a normed space X is a Banach Space (Compllete) (wheather or not X is).

Theorem :- Every finite dimensional normed space is reflexive.

ℓ^p with $1 < p < \infty$ $C[a,b]$

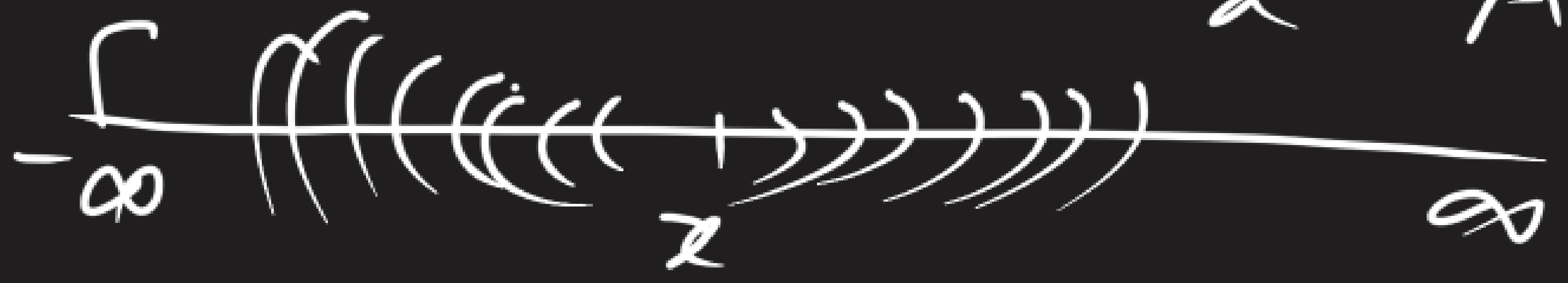
$$((\ell^p)')' = (\ell^q)' = (\ell^p)$$

ℓ^1

ℓ^∞

A $\quad X$

$x \quad A$



$\overline{S} = S \cup S'$

(a,b)

$b-\epsilon \quad b \quad b+\epsilon$

$[a,b]$ $\quad x$

$x \in A \qquad X$

$\mathbb{R} \qquad X$

$\boxed{\overline{S} = S} \longrightarrow$ S dense

(a,b)

$S \longrightarrow S'$

$\boxed{S = S'} \longrightarrow$ S will be closed

if $A \subset X$

if $\exists$ a nbd $G_x \ni$:

$\boxed{x \in G_x \subseteq A}$

$A \subseteq X$

Open/closed

$A^c \longrightarrow$ closed/open

$\mathbb{R} \rightarrow \mathbb{Q} \subset [a,b] \rightarrow ([a,b])' = (a,b) \rightarrow [a,b]$

$A \subset X$

$x_0 \longrightarrow$ l.p.

$\hookrightarrow x_0 \quad A$

$a \rightarrow$ limit pt

then $1 \in \mathbb{R}$

$\mathbb{Q} \subset \mathbb{R}$

$\mathbb{Q}^c \subset \mathbb{Q}$

if for each nbd of x $(a-\epsilon, a+\epsilon)$

excluding x contains

atleast one element.

Open-mapping Theorem :

Lemma. → If B and B' are Banach spaces, and if T is a continuous linear transformation of B onto B', then the image of each open sphere centred on the origin in B contains an open sphere in B' centred on the origin in B'.

Proof:-

We denote by S_r & S_r' the open spheres with radius r centred at origin in B and B'.

It is easy to see that

$$T(S_r) = T(rS_1) = rT(S_1)$$

so it suffices to show that $T(S_1)$ contains some S_r'.

We begin by proving that $\overline{T(S_1)}$ contains some S_r'. Since T is onto we see that $B' = \bigcup_{n=1}^{\infty} T(S_n)$. B' is complete, some $\overline{T(S_n)}$ has an interior point y_0, which may be assumed to lie in $\overline{T(S_{n_0})}$. The mapping $y \to y - y_0$ is a homeomorphism of B' onto itself.

So $\overline{T(S_{n_0})} - y_0$ has the origin as an interior point. Since y_0 is in $T(S_{n_0})$ we have $T(S_{n_0}) - y_0 \subseteq T(S_{2n_0})$. and from this we Obtain

$$\overline{T(S_{n_0})} - y_0 = \overline{T(S_{n_0}) - y_0} \subseteq \overline{T(S_{2n_0})}$$

which shows that the origin is an interior point of $\overline{T(S_{2n_0})}$. Multiplication by any non-zero scalar is a homeomorphism of B' onto itself. So $\overline{T(S_{2n_0})} = \overline{2n_0 T(S_1)}$
$= 2n_0 \overline{T(S_1)}$

and it follows from this that the origin is also an interior point of $\overline{T(S_1)}$, so $S'_\epsilon \subseteq \overline{T(S_1)}$ for some +ve number ϵ.

We conclude the proof by showing that $S'_\epsilon \subseteq T(S_1)$. which is clearly equivalent to $S'_{\epsilon/3} \subseteq T(S_1)$.

Let y be a vector in B' such that

$$\|y\| < \epsilon$$

Since y is in $\overline{T(S_1)}$, $\exists$ a vector x_1 in B such that $\|x_1\| < 1$ and

$$\|y - y_1\| < \epsilon/2, \text{ where } y_1 = T(x_1).$$

We next observe that $S'_{\epsilon/2} \subseteq \overline{T(S_{1/2})}$, so $\exists$ a vector x_2 in B such that $\|x_2\| < 1/2$ and

$$\|(y - y_1) - y_2\| < \epsilon/4, \text{ where } y_2 = T(x_2).$$

Continuing in this way, we obtain a sequence $\{x_n\}$ in B such that

$$\|x_n\| < \frac{1}{2^{n-1}} \text{ and}$$

$$\|y - (y_1 + y_2 + y_3 + \cdots + y_n)\| < \frac{\epsilon}{2^n}$$

where $y_n = T(x_n)$.

If we put

$$S_n = x_1 + x_2 + \cdots + x_n$$

then it follows from $\|x_n\| < 1/2^{n-1}$

that $\{S_n\}$ is a Cauchy sequence in B for which

$$\|S_n\| \leq \|x_1\| + \|x_2\| + \cdots + \|x_n\|.$$

$$< 1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}} < 2.$$

B is complete, so $\exists$ a vector x in B such that $S_n \longrightarrow x$ and $\|x\| = \|\lim S_n\|$

$$= \lim \|S_n\|$$

$$\leq 2 < 3.$$

Shows that x is in S_1. All that remains is to notice that the continuity of T yeilds

$$T(x) = T(\lim S_n)$$

$$= \lim (T(S_n))$$

$$= \lim (y_1 + y_2 + \cdots + y_n)$$

$$= y.$$

From which we see that y is in $T(S_1)$. //