The conjugate of the operator -

Let $T: X \longrightarrow Y$ be a bounded linear operator where X & Y are Normed Spaces. Then the conjugate/adjoint operator $T^*: Y^* \longrightarrow X^*$ of T is defined by

$$f(x) = (T^*(g))x$$

$$= g(Tx).$$

here $g \in Y^*$.

Theorem - (Norm of the adjoint/Conjugate Operator)

The adjoint operator T^* in previous def^n is linear and bounded and

$$\|T^*\| = \|T\|.$$

Proof -

The operator T^* is linear since its domain Y^* is a vector space and we obtain

$$T^*(ag_1 + bg_2)(x) = (ag_1 + bg_2)(Tx)$$

$$= ag_1(Tx) + bg_2(Tx)$$

$$= aT^*(g_1)(x) + bT^*(g_2)(x).$$

Now we will prove our main problem that

$$\|T\| = \|T^*\|.$$

By Previous def^n, we have

$$f = T^*(g) \quad \text{and}$$

by $\|f\| \leq \|g\| \|T\|$ it follows that

$$\|T^*(g)\| = \|f\| \leq \|g\| \|T\|.$$

Taking the sup. over all $g \in Y^*$ of norm one, we obtain the inequality

$$\|T^*\| \leq \|T\|. \quad \text{——— (a)}$$

Hence to prove the remaining part, we must show that

$$\|T\| \leq \|T^*\|.$$

Now by result that "let X be a normed space and $x_0 \neq 0$ be any element of X. then $\exists$ a bounded linear functional $\bar{f}$ on X such that

$$\|\bar{f}\| = 1 \qquad \bar{f}(x_0) = \|x_0\|."$$

implies that for every non-zero $x_0 \in X$ $\exists$ a $g_0 \in Y^*$ such that $\|g_0\| = 1$ & $g_0(Tx_0) = \|Tx_0\|$.

$\Downarrow$

Here $g_0(T(x_0)) = (T^* g_0)(x_0)$ by the def^n of adjoint operator T^* Writing $f_0 = T^* g_0$, we thus obtain

$$\begin{aligned}
\|Tx_0\| &= g_0(T(x_0)) \\
&= f_0(x_0) \\
&\leq \|f_0\| \|x_0\| \\
&= \|T^* g_0\| \|x_0\|. \\
&\leq \|T^*\| \|g_0\| \|x_0\|.
\end{aligned}$$

Since $\|g_0\|=1$, we thus have for every $x_0 \in X$

$$\|Tx_0\| \le \|T^*\| \, \|x_0\|$$

(This includes $x_0 = 0$ since $T(0) = 0$) But

always

$$\|T(x_0)\| \le \underbrace{\|T\|}_{c} \|x_0\|.$$

and here $C = \|T\|$ is the smallest constant C such that

$$\|Tx_0\| \le C\|x_0\|.$$

holds for all $x_0 \in X$. Hence $\|T^*\|$ cannot be smaller that $\|T\|$.

that is $\|T\| \le \|T^*\|$. ——— (b).

From (a) & (b), the theorem is proved.

ie $\|T\| = \|T^*\|$.

$$\|T\| = \sup_{\substack{x \in X \\ \|x\| = 1}} \|Tx\|$$

$$\|T\| = \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Tx\|}{\|x\|}$$