$$\langle \cdot , \cdot \rangle : V \times V \longrightarrow \mathbb{F} \, (\mathbb{R} \mid \mathbb{C})$$

$$\left.\begin{array}{ll}
① & \langle ax+by, z \rangle = a\langle x,z \rangle + b\langle y,z \rangle \\
② & \langle x,y \rangle = \overline{\langle y,x \rangle} \\
③ & \langle x,x \rangle \geq 0 \\
④ & \langle x,x \rangle = 0 \text{ iff } x = 0.
\end{array}\right\}$$

$$\|x\| = \sqrt{\langle x, x \rangle}$$

$$\underline{d(x,y)} = \underline{\|x-y\|} = \underline{\sqrt{\langle x-y, x-y \rangle}}$$

Prove that. $\underline{d(x,y) = d(y,x)}$ $\mathbb{R}$

Prove that.

a) $\langle z, \alpha x + \beta y \rangle$

$$= \bar{\alpha} \langle z, x \rangle + \bar{\beta} \langle z, y \rangle.$$

<u>L.H.S.</u>

$$\langle z, \alpha x + \beta y \rangle = \overline{\langle \alpha x + \beta y, z \rangle}$$

$$= \overline{\alpha \langle x, z \rangle + \beta \langle y, z \rangle}$$

$$= \bar{\alpha} \overline{\langle x, z \rangle} + \bar{\beta} \overline{\langle y, z \rangle}$$

$$= \bar{\alpha} \langle z, x \rangle + \bar{\beta} \langle z, y \rangle$$

Parallelogram Inequality :-

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$$

Proof :-

$$\|x+y\|^2 = \left(\sqrt{\langle (x+y), (x+y) \rangle}\right)^2$$

$$= \langle x+y, x+y \rangle = \langle x, x+y \rangle + \langle y, x+y \rangle$$

$$= \langle x, x \rangle + \langle x, y \rangle + \langle y, y \rangle + \langle y, x \rangle \quad (1)$$

$$= \|x\|^2 + \quad " \quad + \|y\|^2 + "$$

$$\|x-y\|^2 = \|x\|^2 + \|y\|^2 - \langle x,y\rangle - \langle y,x\rangle . \quad \text{—— } 2$$

Adding 1 & 2 we get the result.

$A \perp B$

if $a \perp b$.

$\forall a \in A$

& $\forall b \in B$.

$\langle a,b\rangle = 0$

$\langle x,y\rangle = 0. \quad x \perp y.$

(1) $\mathbb{R}^n$ with

$$\langle x, y \rangle = \xi_1 \eta_1 + \xi_2 \eta_2 + \cdots + \xi_n \eta_n .$$

where $x = (\xi_1, \xi_2, \ldots \xi_n)$

$y = (\eta_1, \eta_2, \ldots \eta_n)$.

$$\|x\| = \sqrt{\langle x, x \rangle}$$

$$\langle x, y \rangle = 0$$

Unitary Space $\mathbb{C}^n$ —

with

$$\langle x, y\rangle = \xi_1\overline{\eta_1} + \xi_2\overline{\eta_2} + \cdots + \xi_n\overline{\eta_n}.$$

$$\|x\| = \left(\sum |x_i|^2\right)^{1/2}$$

$$\langle x, x\rangle = \xi_1\overline{\xi_1} + \xi_2\overline{\xi_2} + \cdots \xi_n\overline{\xi_n}$$

$$\|x\| = \sqrt{\langle x, x\rangle} = \left(|\xi_1|^2 + |\xi_2|^2 + \cdots + |\xi_n|^2\right)^{1/2}$$

Hilbert Sequence Space ℓ^2 —.

$$\boxed{\langle x, y\rangle = \sum_{i=1}^{\infty} \xi_i \overline{\eta_i}}$$

The Space ℓ^p —.

Proof:-

$x = (1, 1, 0, 0, \ldots) \in \ell^p$

$y = (1, -1, 0, 0, \ldots 0) \in \ell^p$

and calculate

$\|x\| = \|y\| = 2^{1/p}$.

& $\|x+y\| = \|x-y\| = 2$.

Space $C[a,b]$ —.

$$x(t) = 1$$

$$y(t) = \frac{t-a}{b-a}.$$

$\|x\| = 1$ & $\|x+y\| = 2$

$\|y\| = 1$ $\quad \|x-y\| = 1.$

Pythagorean Theorem —

If $x \perp y$ in IPS X.

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2$$

$$x \perp y$$

$$\langle x, y \rangle = 0$$

$$\langle y, x \rangle = \overline{\langle x, y \rangle} = \overline{0} = 0.$$

Appolonius Identity:

$$\|z-x\|^2+\|z-y\|^2=\frac{1}{2}\|x-y\|^2+2\left\|z-\frac{1}{2}(x+y)\right\|^2$$

Lemma (Schwarz Inequality, triangle inequality) —.

(a) we have

$\{x, y\}$

$$|\langle x, y\rangle| \leq \|x\|\|y\| \quad (1) \quad \text{(Schwarz Inequality}$$

(b) That norm also satisfies

$$\|x+y\| \leq \|x\| + \|y\|. \qquad \text{(Triangle Inequality)}$$

Proof (a): — If $y=0$ then (1) holds since $\langle x, 0\rangle = 0$. Let $y \neq 0$

for every scalar α we have

$$0 \leq \|x-\alpha y\|^2 = \langle x-\alpha y, x-\alpha y\rangle$$

$$= \langle x,x\rangle - \bar{\alpha}\langle x,y\rangle - \alpha\left[\langle y,x\rangle - \bar{\alpha}\langle y,y\rangle\right].$$

We see that the expression in the brackets is zero if we choose $\bar{\alpha} = \frac{\langle y, x \rangle}{\langle y, y \rangle}$. The remaining inequality is

$$0 \leq \langle x, x \rangle - \frac{\langle y, x \rangle}{\langle y, y \rangle} \langle x, y \rangle = \|x\|^2 - \frac{|\langle x, y \rangle|^2}{\|y\|^2}$$

$$\Rightarrow \quad |\langle x, y \rangle| \leq \|x\| \cdot \|y\|$$

Equality holds in this iff $y \neq 0$

or $0 = ||x - \alpha y||^2$, hence $x - \alpha y = 0$

So that

$$x = \alpha y$$

which shows that x & y are l.d.

(b) We have

$$\|x+y\|^2 = \langle x+y, x+y \rangle$$
$$= \|x\|^2 + \langle x, y \rangle + \langle y, x \rangle + \|y\|^2$$

By Schwarz inequality

$$|\langle x, y \rangle| = |\langle y, x \rangle| \leq \|x\| \|y\| .$$

By triangle inequality for numbers we thus obtain

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle x, y \rangle + \|y\|^2$$
$$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$$
$$= (\|x\| + \|y\|)^2 .$$

Hence

$$\|x+y\| \leq \|x\| + \|y\| .$$

$\langle x, y\rangle + \langle y\, x\rangle = 2\,\|x\|\,\|y\|.$

$\langle x, y\rangle = Re\langle x, y\rangle \geq 0$

by (3)

& $c \geq 0$ follows from

$0 \leq \langle x, y\rangle = \langle cy, y\rangle$

$= c\|y\|^2$

$2\,Re(\langle x, y\rangle)$

Hence $Re(\langle x, y\rangle) = \|x\|\,\|y\| \geq |\langle x, y\rangle|$

(3)