Ques- Reduce the equation

$$yr + (x+y)s + xt = 0$$

to the canonical form.

Solⁿ -

$$S^2 - 4RT = (x+y)^2 - 4yx$$

$$= (x-y)^2 \geq 0$$

$x \neq y \Rightarrow$ hyperbolic

λ-quadratic eqⁿs are

$$R\lambda^2 + S\lambda + T = 0$$

$$y\lambda^2 + (x+y)\lambda + x = 0$$

$$\Rightarrow (y\lambda + x)(\lambda + 1) = 0$$

So that $\lambda = -1$

& $\lambda = -\frac{x}{y}$.

Then the corresponding char. eqⁿ are given by

$$\frac{dy}{dx} - 1 = 0 \qquad \frac{dy}{dx} - \frac{x}{y} = 0.$$

Integrating these

$$y - x = c_1 \quad \& \quad \frac{y^2}{2} - \frac{x^2}{2} = c_2$$

In order to reduce the given PDE to its cannonical form, we choose

$$u = y - x \quad \& \quad v = \frac{y^2}{2} - \frac{x^2}{2}.$$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = -\left(\frac{\partial z}{\partial u} + x \frac{\partial z}{\partial v}\right)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$= \frac{\partial z}{\partial u} + y \frac{\partial z}{\partial v}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

$$= -\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) - \frac{\partial}{\partial x}\left(x \frac{\partial z}{\partial v}\right)$$

$$= -\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) - \left[x \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right) + \frac{\partial z}{\partial v}\right]$$

$$= -\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) - x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right) - \frac{\partial z}{\partial v}$$

$$= -\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\cdot\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\cdot\frac{\partial v}{\partial x}\right] - x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\cdot\frac{\partial v}{\partial x}\right] - \frac{\partial z}{\partial v}$$

$$= -\left[-\frac{\partial^2 z}{\partial u^2} - x\frac{\partial^2 z}{\partial v \partial u}\right] - x\left[-\frac{\partial^2 z}{\partial u \partial v} - x\frac{\partial^2 z}{\partial v^2}\right] - \frac{\partial z}{\partial v}$$

$$r = \frac{\partial^2 z}{\partial u^2} + 2x\frac{\partial^2 z}{\partial u \partial v} + x^2\frac{\partial^2 z}{\partial v^2} - \frac{\partial z}{\partial v}$$

$$t = \frac{\partial^2 z}{\partial u^2} + (2y)\frac{\partial^2 z}{\partial u \partial v} + y^2 \frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial v}$$

$$y\left[\frac{\partial}{\partial u}\frac{\partial z}{\partial v}\cdot\frac{\partial u}{\partial x} + \frac{\partial}{\partial u}\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}\right]$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} + y\frac{\partial z}{\partial v}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{\partial}{\partial x}\left(y\frac{\partial z}{\partial v}\right)$$

$$= \left[\frac{\partial}{\partial u}\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\frac{\partial z}{\partial u}\cdot\frac{\partial v}{\partial x}\right]$$

$$s = -\frac{\partial^2 z}{\partial u^2} - (x+y)\frac{\partial^2 z}{\partial u \partial v} - xy\frac{\partial^2 z}{\partial v^2}$$

Putting values of r, s, t in the given PDE, we get.

$$u^2 \frac{\partial^2 z}{\partial u \partial v} + u \frac{\partial z}{\partial v} = 0$$

$$\boxed{u \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial z}{\partial v} = 0}$$

$$\boxed{u \neq 0}.$$

Working rule for reducing a parabolic equation to its canonical form —

Step-① - Let the given eqⁿ

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \quad \text{—} ①$$

be parabolic so that

$$S^2 - 4RT = 0$$

Step-② - Write λ-quadratic eqⁿ

$$R\lambda^2 + S\lambda + T = 0 \quad \text{—} ②$$

Let λ_1, λ_1 be two equal roots of (2)

Step-③ - Write the char. eqn corresponding to $\lambda = \lambda_1$ ie.

$$\frac{dy}{dx} + \lambda_1 = 0$$

Solving it, we get

$f_1(x,y) = C_1$ where C_1 is constant.

Step-④ :-

Choose $u = f_1(x,y)$

$$v = f_2(x,y) \quad \text{--- (4)}$$

where $f_2(x,y)$ is an arbitrary fn of x & y and is independent of $f_1(x,y)$. For verify this Jacobian J of u & v given by (4) is non zero

ie. $$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \neq 0 \quad \text{--- (5)}$$

Step-(5). Using relation (4),

find p, q, r, s, t in terms of u & v

Step-(6). - Putting these values in (1) we get the required form.

ie.

$$\frac{\partial^2 z}{\partial u^2} = \phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right) \quad \text{or} \quad \frac{\partial^2 z}{\partial v^2} = \phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right).$$

Que:- $\frac{\partial^2 z}{\partial x^2} + 2\left(\frac{\partial^2 z}{\partial x \partial y}\right) + \frac{\partial^2 z}{\partial y^2} = 0$

Reduce the given PDE into its canonical form.