

Lemma - (Continuity of inner product) —

If \mathcal{H} is an inner product space
 $x_n \rightarrow x$ and $y_n \rightarrow y$

then $\langle x_n, y_n \rangle \rightarrow \langle x, y \rangle$

Proof - Subtracting and adding
a term, using the triangle
inequality for numbers and
finally, the Schwarz
inequality, we obtain

$$|\langle x_n, y_n \rangle - \langle x, y \rangle| = |\langle x_n, y_n \rangle - \langle x_n, y \rangle + \langle x_n, y \rangle - \langle x, y \rangle|$$

$$\leq |\langle x_n, y_n - y \rangle| + |\langle x_n - x, y \rangle|.$$

$$\leq \|x_n\| \|y_n - y\| + \|x_n - x\| \|y\|.$$

Since $y_n \rightarrow y$ & $x_n \rightarrow x$

we have

$$|\langle x_n, y_n \rangle - \langle x, y \rangle| \rightarrow 0.$$

Theorem - (Completion) -

For any inner product space X

\exists a Hilbert space H and an isomorphism

A from X into a dense subspace $W \subset H$.

The space H is unique except for isomorphisms.

Proof -

By theorem (*), \exists a Banach space H and an isometry A from X onto a subspace W of H which is dense in H . For reasons of continuity, under such an isometry, sums, scalar multiples of elements in X and W correspond to each other, so that A is even

Theorem - (*) Let $X = (X, \|\cdot\|)$ be a normed space. Then \exists a Banach space \hat{X} and an isometry A from X onto a subspace W of \hat{X} which is dense in \hat{X} . The space \hat{X} is unique, except for isometries.

an isomorphism of X onto W ,
both regarded as normed space.

Previous lemma shows that we
can define an inner product on H
by setting

$$\langle \hat{x}, \hat{y} \rangle = \lim_{n \rightarrow \infty} \langle x_n, y_n \rangle$$

The notation being in Theorem (\star), that is

(x_n) & (y_n) are representations
of $\hat{x} \in H$ & $\hat{y} \in H$, respectively

Hence we see that A is
isomorphism of X onto W ,
both regarded as I.P.S.

Theorem (\star) also guarantees
that H is unique except

for isometries, that is
two completion H & \hat{H} of X

are related by an isometry
 $T: H \rightarrow \hat{H}$. Reasoning as
in the case of A , we conclude
that T must be isomorphism of
the Hilbert space H onto the
Hilbert space \hat{H} .

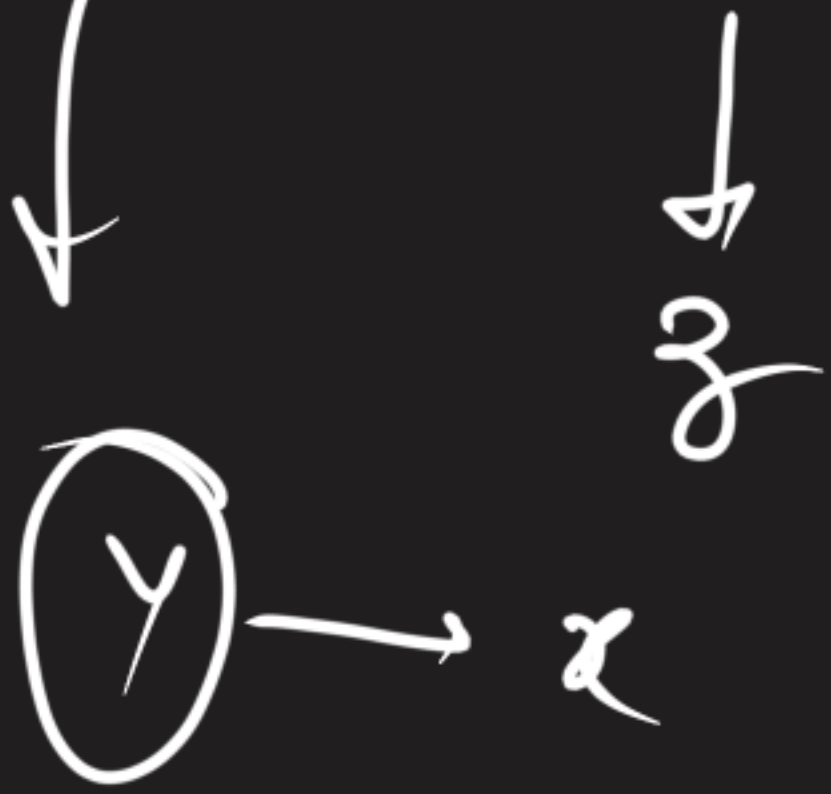
Theorem (Subspace) —

Let Y be a subspace of
a Hilbert space H then

- a) Y is complete iff Y
is closed in H .
- b) $\exists Y$ is finite dimensional
then Y is complete.
- c) $\exists H$ is separable so is Y .

Orthogonal Complements!

$$Y^\perp = \{z \in H : z \perp Y\}$$



$$\langle x, z \rangle = 0$$

$$\forall x \in Y \\ z \in Y^\perp$$