Lemma - (Continuity of inner product) —.

If in an inner product space $x_n \longrightarrow x$ and $y_n \longrightarrow y$

then $\langle x_n, y_n \rangle \longrightarrow \langle x, y \rangle$

Proof. - Substracting and adding a term, using the triangle inequality for numbers and finally, the schwarz inequality, we obtain

$$|\langle x_n, y_n \rangle - \langle x, y \rangle| = |\langle x_n, y_n \rangle - \langle x_n, y \rangle + \langle x_n, y \rangle - \langle x, y \rangle|$$

$$\leq |\langle x_n, y_n - y\rangle| + |\langle x_n - x, y\rangle|.$$

$$\leq \|x_n\| \, \|y_n - y\| + \|x_n - x\| \, \|y\|.$$

Since $y_n \to y$ & $x_n \to x$

we have

$$|\langle x_n, y_n\rangle - \langle x, y\rangle| \longrightarrow 0.$$

Theorem:- (Completion) :-

For any inner product space X $\exists$ a Hilbert space H and an isomorphism A from X into a dense subspace $W \subset H$.

The space H is unique except for isomorphisms.

Proof:-

By theorem (✪), $\exists$ a Banach space H and an isometry A from X onto a subspace W of H which is dense in H. For reasons of continuity, under such an isometry, sums, scalar multiples of elements in X and W correspond to each other, so that A is even

Theorem-(✪) Let $X = (X, \|\cdot\|)$ be a normed space. Then $\exists$ a Banach space $\hat{X}$ and an isometry A from X onto a subspace W of $\hat{X}$ which is dense in $\hat{X}$. The space $\hat{X}$ is unique, except for isometries.

an isomorphism of X onto W, both regarded as normed space.

Previous lemma shows that we can define an innerproduct on H by setting

$$\langle \hat{x}, \hat{y} \rangle = \lim_{n\to\infty} \langle x_n, y_n \rangle$$

the notations being in Theorem(☆), that is (x_n) & (y_n) are representatives of $\hat{x} \in H$ & $\hat{y} \in H$, respectively.

Hence we see that A is isomorphism of X onto W, both regarded as I.P.S.

Theorem(☆) also guarantees that H is unique except for isometries, that is two completion H & $\hat{H}$ of X

are related by an isometry $T : H \to \hat{H}$. Reasoning as in the case of A, we conclude that T must be isomorphism of the Hilbert space H onto the Hilbert space $\hat{H}$.

Theorem (Subspace) —

Let Y be a subspace of a Hilbert space H then

a) Y is complete iff Y is closed in H.

b) If Y is finite dimensional then Y is complete.

c) If H is separable so is Y.

Orthogonal Complements:

$$Y^{\perp} = \{ z \in H : z \perp Y \}.$$

$$Y \longrightarrow x \qquad z$$

$$\boxed{\langle x, z \rangle = 0}$$

$$\boxed{\begin{array}{l} \forall\, x \in Y \\ z \in Y^{\perp}. \end{array}}$$