Monge's Methods —

$$f(x, y, z, p, q, r, s, t) = 0 \quad \text{——— (1)}$$

Type -(1) — When the given eq^n $Rr + Ss + Tt = V$ leads to two distinct intermediate integrals and both of them are used to get the desired sol^n ———.

Step(1) — Write the given eq^n in the standard form $Rr + Ss + Tt = V$

Step-(2) :-

Substitute the values of R, S, T, V in the Monge's subsidiary eq^n

$$R\,dp\,dy + T\,dq\,dx - V\,dx\,dy = 0 \quad \text{--- (1)}$$

$$R(dy)^2 - S\,dx\,dy + T(dx)^2 = 0 \quad \text{--- (2)}$$

Step-(3) :-

Factorise (1) into two distinct factors.

Step-(4) :- Using one of them (factors) obtained in (1) & (2) will lead to an intermidi-ate integral.

Step (5) :-

Solve the two intermidiate integrals obtained in step (4) and get the values of p & q.

Step (6) :- Substitute the value of p & q in $dz = p\,dx + q\,dy$ and integrate to arrive at the required solⁿ.

Ex :-

Solve $r = a^2 t$

Solⁿ :-

Compare with

$$Rr + tT + sS = V$$

$$R = 1,\ T = -a^2,\ S = 0,\ V = 0.$$

Substituting values in M.S.E.

$$R\,dp\,dy + T\,dq\,dx - V\,dx\,dy = 0$$

$$R(dy)^2 - S\,dx\,dy + T(dx)^2 = 0.$$

$$dp\,dy - a^2 dq\,dx = 0 \quad \text{--- (1)}$$

$$(dy)^2 - a^2(dx)^2 = 0 \quad \text{--- (2)}$$

Eqn (2) may be factorised as

$$(dy - a\,dx)(dy + a\,dx) = 0$$

Hence two system of eqn to be considered are

$$dp\,dy - a^2 dq\,dx = 0 \qquad dy - a\,dx = 0 \quad \text{--- (3)}$$

$$dp\,dy - a^2 dq\,dx = 0 \qquad dy + a\,dx = 0 \quad \text{--- (4)}$$

Integrating the second eqn of (3) we get

$$y - ax = c_1 \quad \text{--- (5)}$$

Eliminating $\frac{dy}{dx}$ between the eqns of (3), we get

$$dp - a\,dq = 0$$

so that $p - aq = c_2$ --- (6)

Hence the intermidiate integral corresponding to (3) is

$$p - aq = \phi_1(y - ax) \quad \text{--- (7)}$$

Similarly another intermidiate integral corresponding to (4) is

$$p + aq = \phi_2(y + ax) \quad \text{--- (8)}$$

Solving (7) & (8) for p & q we have

$$p = \frac{1}{2}\left\{\phi_2(y+ax) + \phi_1(y-ax)\right\}$$

$$\& \; q = \frac{1}{2a}\left\{\phi_2(y+ax) - \phi_1(y-ax)\right\}.$$

Substituting these values in

$$dz = p\,dx + q\,dy.$$

$$dz = \frac{1}{2}\left\{\phi_2(y+ax) + \phi_1(y-ax)\right\}dx + \frac{1}{2a}\left\{\phi_2(y+ax) - \phi_1(y-ax)\right\}dy.$$

$$= \frac{1}{2a}\,\phi_2(y+ax)\left(dy + a\,dx\right) - \frac{1}{2a}\,\phi_1(y-ax)\left(dy - a\,dx\right)$$

On integrating, we get

$$\boxed{z = \Psi_2(y+ax) + \Psi_1(y-ax)}$$

Ψ_1 & Ψ_2 are arbitrary functions.

Que. Solve

$$r + (a+b)s + abt = xy.$$

$$y - bx = C_1$$

$$\frac{dy}{dx} = b$$

$$dp\,dy + ab\,dq\,dx - xy\,dx\,dy = 0$$

$$dp + ab\,dq\,\frac{dx}{dy} - x(C_1 + bx)\,dx = 0$$

$$dp + a\,dq - x(C_1 + bx)\,dx = 0$$

$$p + aq - \frac{1}{2}yx^2 + \frac{1}{6}bx^3 = C_2$$