① Shell Balance / General.

② E O C

① E O C

② E O M

x, y, z

r, θ, z

r, θ, ϕ

- Newtonian, modified — μ const
- Incompressible — ρ = const

E O C

E O M

- Fully developed → B O C
- Laminar → x, y, z
- Steady state

Origin — $\frac{dv_z}{dz} = 0$ → F D f, Symmetry

B.C →

axis of symmetry

Application

radial flow

v_r ✓

v_θ ✗

v_z ✗

free surface

$P = P_{atm}$

$\frac{\partial v_r}{\partial r} = 0 = f(\theta, z, \text{const})$

$v_r \neq f(r)$

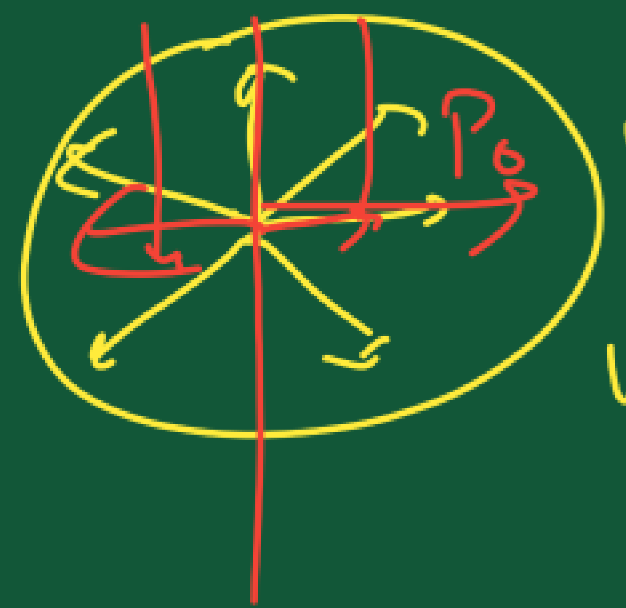
$P_0 > = 0, P_0$

P is geometric

$\frac{\partial P}{\partial r} = 0$ symmetry wrt

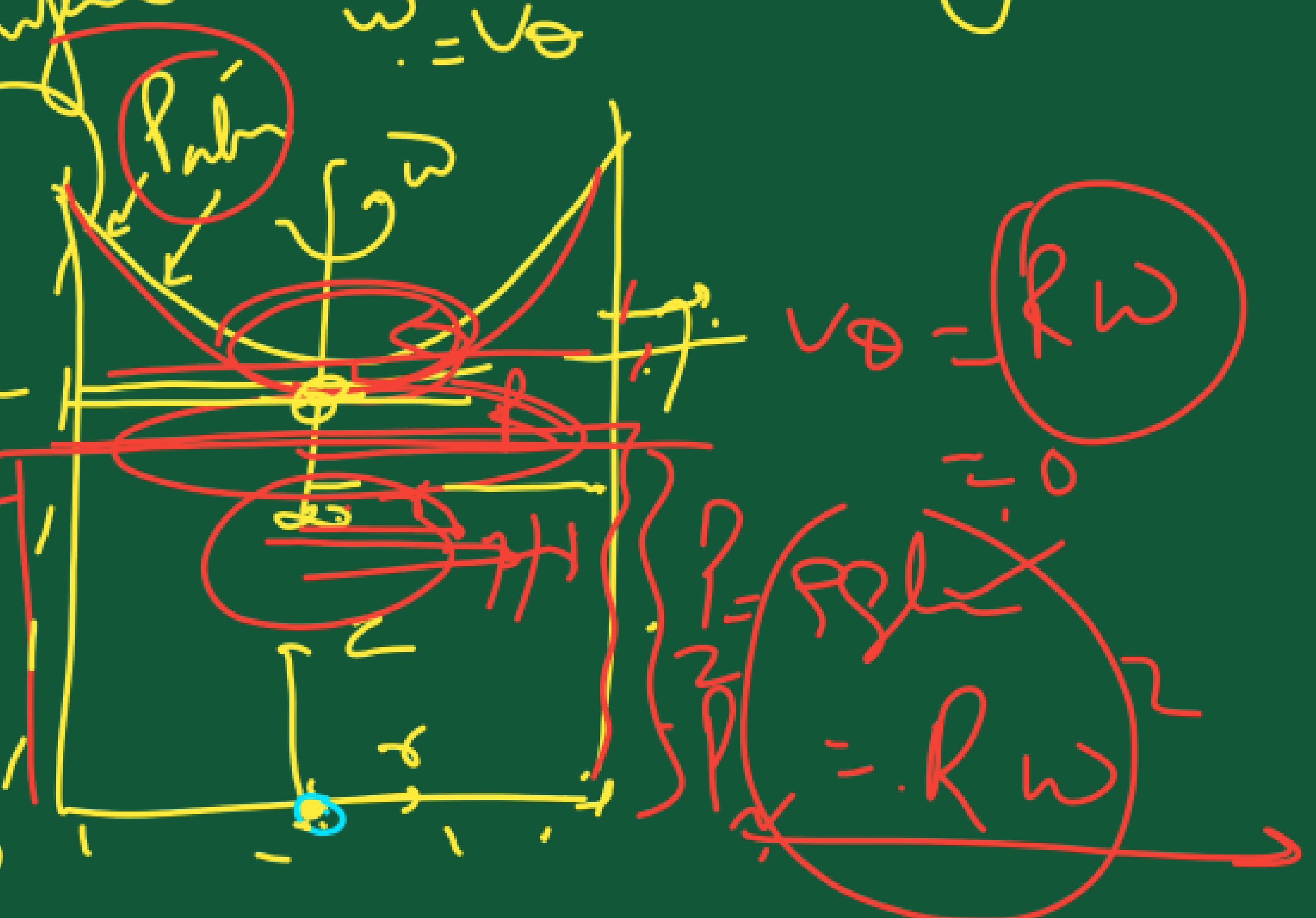
$P \neq f(r)$

$\frac{\partial(\ )}{\partial r}, \frac{\partial(\ )}{\partial \theta}, \frac{\partial(\ )}{\partial z}$



Shape of the surface of rotating fluid

free surface P_{atm} $\omega = v_\theta$



z_0

P_2

$v_\theta = R\omega$

$= 0$

$P = \rho g h$

$= R\omega^2$

ω

$v_\theta = r\omega$

$v_r = 0$

$v_z = 0$

$\rho, \mu = \text{const}$

Steady state

$P \Rightarrow \frac{\partial P}{\partial r}, \frac{\partial P}{\partial \theta}, \frac{\partial P}{\partial z}$

centrifugal force

$v_\theta = r\omega$

$F = \frac{mv^2}{r}$

$P = \frac{F}{A}$

$= mr\omega^2$

EOC $\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0$

S.S

$\frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) = 0 \Rightarrow \frac{\partial}{\partial \theta}(v_\theta) = 0$

FDF in θ dir

$v_\theta \Rightarrow$ symmetry w.r.t. θ

EOM

ωR $v_\theta = r\omega$

$v_\theta \neq f(\theta)$

$v_\theta = f(r, z, \text{const})$

fixed surface

θ at $r = R$

$v_\theta = 0$

drag

dived

$\boxed{v_\theta \neq f(z)}$

$\frac{\partial v_\theta}{\partial z} = 0$

$\boxed{v_\theta = f(r)}$

$\frac{\partial^2 v_\theta}{\partial z^2} = 0$

r comp.

$$\rho\left(\cancel{\frac{\partial v_r}{\partial t}} + v_r\cancel{\frac{\partial v_r}{\partial r}} + \frac{v_\theta}{r}\cancel{\frac{\partial v_r}{\partial \theta}} - \frac{v_\theta^2}{r} + v_z\cancel{\frac{\partial v_r}{\partial z}}\right) = -\frac{\partial P}{\partial r} + \rho \cancel{g_r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\cancel{(r v_r)}\right) + \frac{1}{r^2}\cancel{\frac{\partial^2 v_r}{\partial \theta^2}} - \frac{2}{r^2}\cancel{\frac{\partial v_\theta}{\partial \theta}} + \cancel{\frac{\partial^2 v_r}{\partial z^2}}\right]$$

S.S $\quad v_z=0 \quad v_r=0 \quad v_r=0 \quad g_r=0 \quad v_r=0 \quad v_r=0 \quad$ F.O.C $\quad v_r=0$

$$-\frac{\rho v_\theta^2}{r} = -\frac{\partial P}{\partial r} \quad \text{—— (1)}$$

z comp

$$\rho\left(\cancel{\frac{\partial v_z}{\partial t}} + v_r\cancel{\frac{\partial v_z}{\partial r}} + \frac{v_\theta}{r}\cancel{\frac{\partial v_z}{\partial \theta}} + v_z\cancel{\frac{\partial v_z}{\partial z}}\right) = -\frac{\partial P}{\partial z} + \rho g_z + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\cancel{\left(r\frac{\partial v_z}{\partial r}\right)} + \frac{1}{r^2}\cancel{\frac{\partial^2 v_z}{\partial \theta^2}} + \cancel{\frac{\partial^2 v_z}{\partial z^2}}\right)$$

S.S $\quad v_r=0,\ v_z=0 \quad v_z=0 \quad v_z=0 \quad v_z=0 \quad v_z=0 \quad v_z=0$

$$-\frac{\partial P}{\partial z} + \rho g_z = 0 = P = \rho g z$$

Main θ comp.

$$\rho\left(\cancel{\frac{\partial v_\theta}{\partial t}} + v_r\cancel{\frac{\partial v_\theta}{\partial r}} + \frac{v_\theta}{r}\cancel{\frac{\partial v_\theta}{\partial \theta}} + \cancel{\frac{v_r v_\theta}{r}} + v_z\cancel{\frac{\partial v_\theta}{\partial z}}\right) = -\cancel{\frac{1}{r}\frac{\partial P}{\partial \theta}} + \rho \cancel{g_\theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) + \frac{1}{r^2}\cancel{\frac{\partial^2 v_\theta}{\partial \theta^2}} + \frac{2}{r^2}\cancel{\frac{\partial v_r}{\partial \theta}} + \cancel{\frac{\partial^2 v_\theta}{\partial z^2}}\right]$$

S.S $\quad v_r=0 \quad$ F.O.C $\quad v_r=0 \quad v_z=0 \quad 0 \quad 0 \quad$ F.O.C $\quad v_r=0 \quad$ F.O.C

$\boxed{V_\theta = r\omega}$

$r\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r V_\theta)\right)\right] = 0$

$\frac{1}{r}\frac{\partial}{\partial r}(r V_\theta) = C_1$

$\int \frac{\partial}{\partial r}(r V_\theta) = \int C_1 r \, dr$

$r V_\theta = C_1 \frac{r^2}{2} + C_2$

$\boxed{V_\theta = \frac{C_1 r}{2} + \frac{C_2}{r}} \rightarrow \infty$

B.C

$\frac{\partial^2 V_\theta}{\partial z^2} = 0$

B.C

$r = R, \; V_\theta = R\omega$

$r = 0, \; V_\theta = \text{finite}$

$r = 0$ finite $\Rightarrow C_2 = 0$

$V_\theta = \frac{C_1 r}{2}$

$r = R$: $\quad R\omega = \frac{C_1 R}{2}$

$C_1 = 2\omega \Rightarrow \boxed{V_\theta = \omega r}$

$P_{total} \Rightarrow$

$dP = \frac{\partial P}{\partial r} dr + \cancel{\frac{\partial P}{\partial \theta} d\theta} + \frac{\partial P}{\partial z} dz$

$\int_{P_2}^{P_1} dP = \frac{\partial P}{\partial r} dr + \frac{\partial P}{\partial z} dz$

$= \int_0^R \frac{\rho V_\theta^2}{r} dr + \int_{z_0}^{z} \rho g \, dz$

$\int \frac{\rho r^{\cancel{2}} \omega^2}{\cancel{r}}$

$P_1 = P_{atm}$

$P_1 - P_2 = \left. \frac{\rho \omega^2 r^2}{2} \right|_0^R + \rho g z \Big|_{z_0}^{z}$

$P_2 - P_1 \quad \rho g (z - z_0)$

$$P_2 - P_1 = -\frac{\rho\omega^2 r^2}{2} + \rho g(z-z_0)$$

$P_{atm} - P_{atm}$

$$= \frac{\cancel{\rho}\,\omega^2 r^2}{2} = \cancel{\rho}\, g(z-z_0)$$

for free surface.

$P = P_{atm}$

$$\boxed{z = \frac{\omega^2 r^2}{2g} + z_0}$$

condition

$\boxed{P = P_{atm}}$

$P_1 = P_{atm}$

$P = f(r,z)$

$\downarrow$

$P = P_{atm}$