Electromagnetic wave equation

Let us consider a uniform linear medium having permittivity (∈), permeability (μ), and conductivity (σ), but not any charge or current other than that determined by Ohm's law. Then, $D = \in E$, $B = \mu H$, $J = \sigma E$ and ρ=0. D is electric displacement field (electric flux density) and H is magnetic field strength.

We know Maxwell's equation-

$$Div\vec{E} = 0 \qquad (1)$$

$$Div\vec{H} = 0 \qquad (2)$$

$$Curl\,\vec{\mathrm{E}} = -\mu\frac{\partial\vec{\mathrm{H}}}{\partial t} \qquad (3)$$

$$curl\,\vec{\mathrm{H}} = \sigma\vec{E} + \epsilon\,\frac{\partial\vec{E}}{\partial t} \qquad (4)$$

Taking the curl of equation (3),

$$Curl\;Curl\,\vec{\mathrm{E}} = -\mu\frac{\partial\,(\mathrm{curl}\,\vec{\mathrm{H}}\,)}{\partial t}$$

Putting the value of $curl\,\vec{\mathrm{H}}$ from (4) than we get-

$$Curl\;Curl\,\vec{\mathrm{E}} = -\mu\frac{\partial}{\partial t}(\sigma\vec{E} + \epsilon\,\frac{\partial\vec{E}}{\partial t}) = -\sigma\,\mu\frac{\partial\vec{E}}{\partial t} - \epsilon\mu\frac{\partial^2\vec{E}}{\partial t^2} \qquad (5)$$

Similarly, taking the curl of equation (4), putting the value of $curl\,\vec{\mathrm{E}}$ from (3) than we get-

$$Curl\;Curl\,\vec{\mathrm{H}} = -\sigma\,\mu\frac{\partial\vec{H}}{\partial t} - \epsilon\mu\frac{\partial^2\vec{H}}{\partial t^2} \qquad (6)$$

Now using vector identity, $url\;Curl\,\vec{\mathrm{A}} = grad\;div\,\vec{\mathrm{A}} - \nabla^2\vec{\mathrm{A}}$, and keeping view equation (1) and (2), than equation (5) and (6) will be—

$$\nabla^2\vec{\mathrm{E}} - \sigma\,\mu\frac{\partial\vec{E}}{\partial t} - \epsilon\mu\frac{\partial^2\vec{E}}{\partial t^2} = 0 \qquad (7)$$

$$\nabla^2\vec{\mathrm{H}} - \sigma\,\mu\frac{\partial\vec{H}}{\partial t} - \epsilon\mu\frac{\partial^2\vec{H}}{\partial t^2} = 0 \qquad (8)$$

Equation (7) and (8) represent wave equation which govern the E.M. field in a homogeneous, linear medium in which the charge density is zero.

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Electromagnetic wave In Vaccum

For free space $\in = 0, \mu = 0, \ \sigma = 0$ and ρ=0, than Maxwell's equation will be-

$Div\vec{E} = 0$ (1)

$Curl\,\vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$ (3)

$Div\vec{H} = 0$ (2)

$curl\,\vec{H} = \epsilon_0 \ \frac{\partial \vec{E}}{\partial t}$ (4)

Taking the curl of equation (3),

$$Curl\ Curl\,\vec{E} = -\mu_0 \frac{\partial\,(\text{curl}\,\vec{H}\,)}{\partial t}$$

Putting the value of $curl\,\vec{H}$ from (4) than we get-

$$Curl\ Curl\,\vec{E} = -\mu_0\ \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \quad (5)$$

Similarly, taking the curl of equation (4), putting the value of $curl\,\vec{E}$ from (3) than we get-

$$Curl\ Curl\,\vec{H} = -\mu_0\ \epsilon_0 \frac{\partial^2 \vec{H}}{\partial t^2} \quad (6)$$

Now using vector identity, $url\ Curl\,\vec{A} = grad\ div\,\vec{A} - \nabla^2\vec{A}$, and keeping view equation (1) and (2), than equation (5) and (6) will be—

$$\nabla^2\vec{E} - \mu_0\ \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \quad (7)$$

$$\nabla^2\vec{H} - \mu_0\ \epsilon_0 \frac{\partial^2 \vec{H}}{\partial t^2} = 0 \quad (8)$$

Equation (7) and (8) will be representing wave equation governing electro-magnetic field ($\vec{E}$ and $\vec{H}$) in free space. Equation (7) and (8) are similar to scalar wave equation-

$$\nabla^2 U - \mu_0\ \epsilon_0 \frac{\partial^2 U}{\partial t^2} = 0 \quad (9)$$

Here U is scalar and can sand for one component of $\vec{E}$ and $\vec{H}$ equation (9) resemble with general wave equation-

$$\nabla^2 U = \frac{1}{v^2}\ \frac{\partial^2 U}{\partial t^2} \quad (10)$$

Where v is velocity of wave, Comparing equation (9) and (10), we use field vector $\vec{E}$ and $\vec{H}$ are propagate in free space as waves at a speed equal to-

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$$v = \frac{1}{\sqrt{\mu_0\,\epsilon_0}} = \sqrt{\frac{4\pi}{4\pi\mu_0\,\epsilon_0}} = \sqrt{\frac{4\pi\times 9\times 10^{9}}{4\pi\times 10^{-7}}} = 3\text{x}10^{8}\ \text{m/s} \qquad (11)$$

Which is velocity of light (C).Therefore, it is reasonable to write C the speed of light in place of $\frac{1}{\sqrt{\mu_0\,\epsilon_0}}$, so equation (7), (8) and (9) will be-

$$\nabla^2\vec{\mathrm{E}} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0$$

$$\nabla^2\vec{\mathrm{H}} - \frac{1}{c^2}\frac{\partial^2\vec{H}}{\partial t^2} = 0 \qquad (12)$$

$$\nabla^2 U - \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} = 0$$