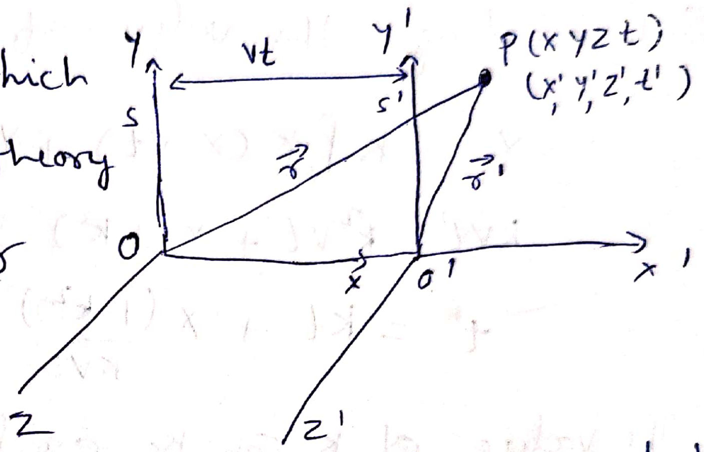
Lorentz Transformation

The transformation equation which based on postulats of special theory of Relativity is called Lorentz Transformation equation.



Consider two inertial frame S and S' with origin o and o' respectively. Let S' is moving with velocity v relative to S along positive direction of x-axis. Let two observer O and O' observ any event P from system of S and S' respectively. If the co-ordinate of two inertial system coincide at $t = t' = 0$ then the event P is determined by the co-ordinate (x, y, z, t) and (x', y', z', t') by observers O and O' respectively.

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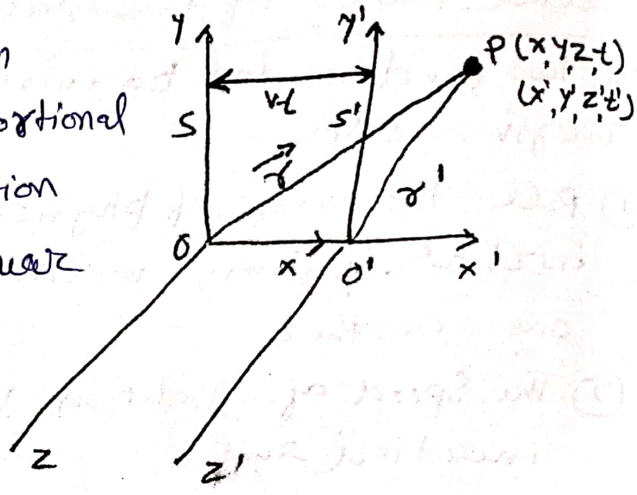

The measurment in the x direction made in frame 'S' should be proportional to that made in S', i.e. the equation between X and X' must be linear and be of the form.

$$x' = k(x - vt) \quad \text{--- (1)}$$

$$x = k(x' + vt') \quad \text{--- (2)}$$

Since relative motion of S and S' only along $x \to x'$ axis. then,

$$y = y' \quad \text{and} \quad z = z' \quad (3)$$

but time Co-ordinate t and t' are not equal.

Substituting the value of x' from (1) to (2), we get

$$x = k[k(x - vt) + vt'] = k^2(x - vt) + kvt'$$

$$kvt' = k^2vt + x(1 - k^2)$$

$$t' = kt + x\frac{(1-k^2)}{kv} \quad \text{---- (4)}$$

- The value of k can be evaluated from IInd Postulate, For it, Let a Signal of light is emitted from the common origin of S and S' at time $t = t' = 0$

The Signal travel with speed "c" which is the same for both frames [2nd postulate]. After some time the position of the signal, as seen from S and S' is given by

$$x = ct \quad \text{-- (5)} \quad \text{and} \quad x' = ct' \quad \text{- (6)}$$

Substituting the value of x' and t' from eq (1) and (4) in eq (6) then -

$$k(x-vt) = c\left[kt + \frac{(1-k^2)}{kv}\cdot(x)\right]$$

$$kx - kvt = \left[ckt + \frac{(1-k^2)}{kv}cx\right]$$

$$kx - \frac{(1-k^2)}{kv}cx = ckt + kvt$$

$$x\left[k - \frac{(1-k^2)}{kv}c\right] = \underline{ckt + kvt}$$

$$x = \frac{ckt + kvt}{\left[k - \frac{(1-k^2)}{kv}c\right]}$$

$$x = \frac{ckt\left[1+\frac{v}{c}\right]}{k\left[1-\left(\frac{1-k^2}{k^2}\right)\frac{c}{v}\right]} = \frac{ct\left(1+\frac{v}{c}\right)}{1-\left(\frac{1}{k^2}-1\right)\frac{c}{v}} \quad \text{—(7)}$$

According to eq (5) and (7)

$$ct = \frac{ct\left(1+\frac{v}{c}\right)}{1-\left(\frac{1}{k^2}-1\right)\frac{c}{v}}$$

$$1+\frac{v}{c} = 1 - \frac{1}{k^2}\frac{c}{v} + \frac{c}{v}$$

$$\frac{1}{k^2}\frac{c}{v} = 1 + \frac{c}{v} - 1 - \frac{v}{c} = \frac{c^2-v^2}{cv}$$

$$\frac{1}{k^2} = \left(\frac{c^2-v^2}{cv}\right)\times\left(\frac{v}{c}\right) = \frac{c^2-v^2}{c^2}$$

$$k^2 = \frac{c^2}{c^2-v^2} \quad \text{or} \quad \boxed{k = \frac{c}{\sqrt{c^2-v^2}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}} \quad \text{--(8)}$$

Inserting the value of k in eq (1) we get

$$\boxed{x' = \frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}}} \quad \text{—— (9)}$$

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Again substituting the value of K in eq (4) we have

$$t' = kt + \left(\frac{1-k^2}{kv}\right)x = k\left[t + \left(\frac{k^2-1}{k^2}\right)\frac{x}{v}\right]$$

$$= k\left[t - \left(1 - \frac{1}{k^2}\right)\frac{x}{v}\right] = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left[t - \left\{1 - \left(1-\frac{v^2}{c^2}\right)\right\}\frac{x}{v}\right]$$

$$\boxed{t' = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(t - \frac{vx}{c^2}\right)} \quad -(10)$$

Eq (3) (9) (10) are known as Lorentz transformation equation [from S to S'] they are

$$\boxed{x' = \frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}}, \quad y' = y, \quad z' = z, \quad t' = \frac{t-\frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}} \quad --(11)$$

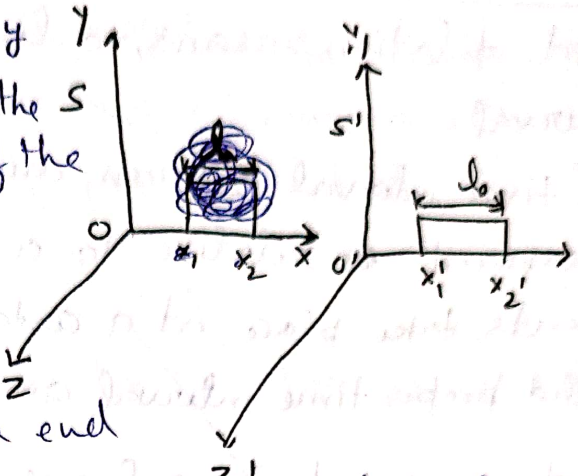
The inverse Lorentz transformation (from S' to S) can be obtained by replacing v by −v and interchanged primed and unprimed condition. So

$$\boxed{x = \frac{x'+vt}{\sqrt{1-\frac{v^2}{c^2}}} \quad y = y' \quad z = z' \text{ and } t = \frac{t'+\frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}} \quad -(12)$$

→

Length Contraction [Proper length]

The length l_0 has been measured by a stationary observer relative to the rod and is called proper length of the rod.



$$l_0 = x_2' - x_1' \quad (1)$$

Now if the x-coordinate of the end points of the rod in frame S are measured to be x_1 and x_2 at the same time t, then in this frame the observed length of rod is

$$l = x_2 - x_1 \quad (2)$$

According to Lorentz transformation

$$x_1' = k(x_1 - vt)$$
$$x_2' = k(x_2 - vt)$$

So $x_2' - x_1' = k(x_2 - x_1)$ or $l_0 = kl$

$$\boxed{l_0 = \frac{l}{\sqrt{1-\frac{v^2}{c^2}}}} \quad \text{or} \quad \boxed{l = l_0\sqrt{1-\frac{v^2}{c^2}}} \quad (3)$$

Since factor $\sqrt{1-\frac{v^2}{c^2}}$ is smaller than unity, we have $l < l_0$. This means that the length of the rod (l) as measured by an observer relative to which the rod is in motion, is smaller than its proper length.

Time Dilation

Consider two reference system S and S'. Let S' moving with a velocity V with respect to S in positive direction of X-axis. Suppose a clock is situated in the in the S frame, at position x and gives signal at interval Δt. Then

$$\Delta t = t_2 - t_1 \quad \cdots \quad (1)$$

if this interval is observed by an observer in S' frame, the interval $\Delta t'$ is

$$\Delta t' = t_2' - t_1' \quad (2)$$

from Lorentz transformation.

$$t_1 = \frac{t_1' + \left(\frac{Vx'}{c^2}\right)}{\sqrt{1-\frac{v^2}{c^2}}} \quad \text{and} \quad t_2 = \frac{t_2' + \left(\frac{Vx'}{c^2}\right)}{\sqrt{1-\frac{v^2}{c^2}}} \quad (3)$$

Putting the value of t_1 and t_2 in eq (1)

$$\Delta t = \frac{t_2' + \left(\frac{Vx'}{c^2}\right)}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{t_1' + \left(\frac{Vx'}{c^2}\right)}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Delta t = \frac{t_2' - t_1'}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{\Delta t'}{\sqrt{1-\frac{v^2}{c^2}}} \quad - (4)$$

This equation shows that $\Delta t > \Delta t'$, that is the time interval in frame S is greater than the time interval in frame S'.

→ If $V << c$ then, $\frac{v^2}{c^2}$ = negligible, then $\Delta t = \Delta t'$

So the time recorded by a moving clock is the same as that when it at rest, which is the case of event observed in daily life.

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→ If $v \simeq c$ then $\sqrt{1-\frac{v^2}{c^2}} \leq 1$ Then $\Delta t > \Delta t'$

So the moving clock shows the time which is dilated at high speed and observed that clock goes slow at high speed.

→ in eq (4) the factor $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ is greater than 1 so

$\Delta t > \Delta t'$ or

So the interval measured in frame S is longer than the time interval in the frame S' in which the events are occuring at a certain point X'. This effect is called "Time dilation". This means that to a stationary observer the moving clock will appear to go slow.

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Relativistic energy [Mass-energy relation]

Mass energy relation of relativity shows the relation between mass and energy.

According to Newton's law of motion, $F = \frac{dP}{dt} = \frac{d(mv)}{dt}$ ①

Since relativistic mechanics, mass m is also variable, so

$$F = m\frac{dv}{dt} + v\frac{dm}{dt} \quad -②$$

From work energy theorem, $dW = dK = F.ds = m\frac{dv}{dt}\cdot ds + v\frac{dm}{dt}ds$

$$= m\frac{ds}{dt}dv + v\frac{ds}{dt}dm = mvdv + v^2dm \quad ---③$$

Now according to theory of relativity, the mass m of a body moving with velocity v is given as, $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$ ④

where m_0 is the rest mass of the body and c is velocity of light

squaring eq ④ $m^2 = \frac{m_0^2}{1-\frac{v^2}{c^2}}$ or $m^2c^2 - m^2v^2 = m_0^2c^2$

Differentiating this equation and keeping in mind that both m_0 and c constant we get

$$2mdmc^2 - v^2 2mdm - m^2 . 2vdv = 0$$

$$mvdv + v^2dm = c^2dm \quad - ⑤$$

Comparing eq ③ with ⑤ we have $dK = c^2dm$

Let the change in K.E. of the body be K when mass of the body changes from m_0 to m. then, $K = \int dM = \int_{m_0}^{m} c^2 dm = c^2(m - m_0)$ ⑥

- Rest mass m_0 is associated with an amount of energy m_0c^2 which is called rest energy of the body. The total energy of the body E = rest mass energy (E_0) + Kinetic energy (K)

$$= m_0c^2 + (m - m_0)c^2$$

$$\boxed{E = mc^2} \quad - ⑦$$

which is known as mass-energy relation. and states a universal equivalence between mass and energy